A q-QUEENS PROBLEM
II. THE SQUARE BOARD

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Abstract. We apply to the $n \times n$ chessboard the counting theory from Part I for nonattacking placements of chess pieces with unbounded straight-line moves, such as the queen. Part I showed that the number of ways to place q identical nonattacking pieces is given by a quasipolynomial function of n of degree $2q$, whose coefficients are (essentially) polynomials in q that depend cyclically on n.

Here we study the periods of the quasipolynomial and its coefficients, which are bounded by functions, not well understood, of the piece's move directions, and we develop exact formulas for the very highest coefficients. The coefficients of the three highest powers of n do not vary with n. On the other hand, we present simple pieces for which the fourth coefficient varies periodically. We develop detailed properties of counting quasipolynomials that will be applied in sequels to partial queens, whose moves are subsets of those of the queen, and the nightrider, whose moves are extended knight's moves.

We conclude with the first, though strange, formula for the classical n-Queens Problem. We state several conjectures and open problems.

Contents

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1. Introduction

The well known n-Queens Problem asks for the number ways to place n nonattacking queens on an $n \times n$ chessboard. No general formula is known; the answer has been computed separately for each small value of n. In this article, the second of a series [2], we treat a natural generalization, the q-Queens Problem, in which both the number of queens, q, and the size of the board, n, vary independently. In Part I we developed a general theory for arbitrary convex-polygonal boards with rational vertices and any chess piece $\mathbb{P}$ with unbounded straight-line moves (known as a "rider"); this includes the queen, rook, and bishop as well as fairy chess pieces such as the nightrider, whose moves are those of a knight extended to any distance. The main result of Part I (Theorem I.4.1) is that the answer is a quasipolynomial function of n, which means it is given by a cyclically repeating sequence of polynomials, and the coefficients of powers of n are essentially polynomial functions of q; in fact, we found a complicated general coefficient formula. We deduced this from the theory of inside-out polytopes, which is an extension of Ehrhart's theory of counting lattice points in convex polytopes.

Here and in later parts we treat the square board. The self-similarity of its interior lattice points permits us to find explicit formulas for the two highest coefficients and partial formulas for the others, for arbitrary riders. Setting $q = n$ we obtain the first known formula for the n-Queens Problem.

In Parts III through V we specialize to specific pieces. Part III applies the theory of Parts I and II to "partial queens", which have a subset of the queen's moves. We found partial queens easier to work with than other pieces with equally many moves; we also think they make good test cases for conjectures. The main examples of partial queens are the bishop, queen, and rook, which we treat in detail along with the nightrider in Parts IV and (for the bishop) V.

Part I serves as a general introduction to the series. Its introduction gives a fuller description of the background of our research, including the valuable hints obtained from the formulas collected and developed by Vaclav Kotěšovec (see [3]). We assume terminology and notation from Part I; we provide a dictionary of notation to assist the reader (and authors).

Briefly, the board consists of the integral points in the dilate $(n+1)\mathcal{B}^\circ = (n+1)(0,1)^2$ of the open unit square. The associated polytope is $\mathcal{P} = \mathcal{B}^q = [0,1]^{2q}$ with $\mathcal{P}^\circ = \mathcal{B}^{\circ q} = (0,1)^{2q}$ and the inside-out polytope is $([0,1]^{2q}, \mathscr{A}_\mathbb{P})$ where $\mathscr{A}_\mathbb{P}$ is an arrangement of hyperplanes determined by the moves of the piece $\mathbb{P}$. We adopt the concise notation $[n] := \{1, \ldots, n\}$ so that the set of points representing the squares of an $n \times n$ chessboard is

$$[n]^2 = (n+1)(0,1)^2 \cap \mathbb{Z}^2.$$

The moves of $\mathbb{P}$ are all the integral multiples of vectors in a nonempty set $\mathbf{M}$ of non-zero, non-parallel integral vectors $m_r = (c_r, d_r) \in \mathbb{R}^2$ reduced to lowest terms (that is, c_r and d_r are relatively prime). The counting function $u_\mathbb{P}(q;n)$ is defined as the number of nonattacking positions of q indistinguishable copies of $\mathbb{P}$ on the $n \times n$ board, and $o_\mathbb{P}(q;n)$ is the number of such positions of q distinguishable copies. By Theorem I.4.1 $u_\mathbb{P}(q;n)$ is a quasipolynomial function of n, which we expand as

$$u_\mathbb{P}(q;n) = \gamma_0(n)n^{2q} + \gamma_1(n)n^{2q-1} + \gamma_2(n)n^{2q-2} + \cdots + \gamma_{2q}(n)n^0.$$

By Ehrhart theory the leading coefficient is $\gamma_0(n) = 1/q!$ and the period of $u_\mathbb{P}(q;n)$ is a divisor of the denominator $D([0,1]^{2q}, \mathscr{A}_\mathbb{P})$, defined as the least common denominator of all coordinates of all vertices of $([0,1]^{2q}, \mathscr{A}_{\mathbb{P}})$. By Theorem I.5.3 the number of unlabelled combinatorial types of nonattacking configuration equals $u_{\mathbb{P}}(q;-1)$.

A trivial observation is that $u_{\mathbb{P}}(1;n) = n^2$ for any piece, and with one piece there is (of course) one combinatorial type. Theorem 2.3 gives a complete solution for $u_{\mathbb{P}}(2;n)$, the counting function for two copies of an arbitrary rider piece. General formulas for $u_{\mathbb{P}}(q;n)$ when $q \geq 3$ are difficult.

A computational approach to finding $u_{\mathbb{P}}(q;n)$ explicitly for a particular piece is to evaluate it at enough small values of n by counting the non-attacking configurations, bounding the period p somehow (possibly by bounding the denominator $D(\mathcal{P}, \mathscr{A}_{\mathbb{P}})$), and using that information to interpolate the coefficients of the p constituent polynomials. To get a confirmed answer, $2pq$ values of $u_{\mathbb{P}}(q;n)$ must be computed. This method becomes hard for most problems because it involves a daunting amount of computation if the period or its best known bound is large, as is usually the case. That is why we think it important to find good bounds on the period.

Any preliminary information about $u_{\mathbb{P}}(q;n)$ can reduce the number of required values. Our main result, Theorem 4.2, reduces that number by $2p-1$ by giving a simple formula for the second coefficient, γ_1, and proving that γ_2 is independent of n. (We reduce it by two more by noting that $u_{\mathbb{P}}(q;0) = 0$ and $u_{\mathbb{P}}(q;1) = \delta_{q1}$, the Kronecker delta.) The proof of Theorem 4.2 depends on the structure of the subspace Ehrhart functions developed in Section 3. In Section 5, through an explicit construction involving pieces with one move direction, we show that γ_3 may depend on n, in contrast to the constancy of γ_0, γ_1, and γ_2.

Setting n and q equal in Theorem 4.2 gives the first known formula for the number of solutions to the n-Queens Problem (Section 6). It is not clear how practical this formula is, as it has infinitely many terms though only finitely many for each n, but it is precise and complete and shows a clear structure from which it may be possible to deduce interesting consequences—which we do not attempt here.

We conclude with several open problems, of which one, dealing with recurrences satisfied by $u_{\mathbb{P}}(q;n)$ for fixed q (Section 7.2), promises to be superbly important.

2. Two Pieces

We examine an exceedingly small number of pieces, i.e., $q = 2$. There is a (relatively) simple way to calculate $u_{\mathbb{P}}(2;n)$. Define $a_{\mathbb{P}}(2;n)$ to be the number of attacking configurations of two labelled pieces $\mathbb{P}$ (which may occupy the same position; that is considered attacking). Then

$$u_{\mathbb{P}}(2;n) = \frac{1}{2!}o_{\mathbb{P}}(2;n) = \frac{1}{2}\big[n^4 - a_{\mathbb{P}}(2;n)\big]. \tag{2.1}$$

Finding $a_{\mathbb{P}}(2;n)$ is easy in principle although nontrivial in detail. (See Equation (2.7).) To begin with, consider a move $(c,d) \in \mathbf{M}$, whose slope is the rational fraction d/c, and let

$$l^{d/c}(b) := \text{the line in } \mathbb{R}^2 \text{ with slope } d/c \text{ and } y\text{-intercept } b.$$

We allow $d/c = 1/0 = \infty$, in which case b is instead the x-intercept. Define

$$l^{d/c}_{\mathcal{B}}(b) := l^{d/c}(b) \cap [n]^2,$$

the set of positions on the $n \times n$ board $[n]^2$ that lie on the line $l^{d/c}(b)$. The multiset of line sizes,

$$\mathbf{L}^{d/c}(n) := \big\{|l^{d/c}_{\mathcal{B}}(b)| : l^{d/c}_{\mathcal{B}}(b) \neq \varnothing\big\},$$

is finite and the sum of its entries is n^2. We need to know the the exact contents of $\mathbf{L}^{d/c}(n)$. Two cases are elementary:

$$\mathbf{L}^{0/1}(n) = \mathbf{L}^{1/0}(n) = \{n^n\},$$
$$\mathbf{L}^{1/1}(n) = \mathbf{L}^{-1/1}(n) = \{1^2, 2^2, \ldots, (n-1)^2, n^1\}.$$

Lemma 2.1. *Assume $0 < c \leq d$ are relatively prime integers. Let $\bar{n} := (n \bmod d)$. The multiplicities of line sizes in $\mathbf{L}^{d/c}(n)$ are as in the following table:*

Line size	$1 \leq l < \lfloor \frac{n}{d} \rfloor$	$\lfloor \frac{n}{d} \rfloor$	$\lfloor \frac{n}{d} \rfloor + 1$
Multiplicity	$2cd$	$(d-\bar{n})(n - c\lfloor \frac{n}{d} \rfloor) + c(\bar{n}+d)$	$\bar{n}(n - c\lfloor \frac{n}{d} \rfloor)$

Proof. Let $\delta := \lfloor n/d \rfloor = (n-\bar{n})/d$; note that $\delta \leq \lfloor n/c \rfloor$.

Each nonempty line $l^{d/c}_{\mathcal{B}}(b)$ has a lowest point

$$(x,y) \in Z := \{(x,y) \in [n]^2 : x \leq c \text{ or } y \leq d\},$$

and conversely, each point in Z is the lowest point of a different line $l^{d/c}_{\mathcal{B}}(b)$. If we rename the line $l_{\mathcal{B}}(x,y)$, the naming is unique and the points on $l_{\mathcal{B}}(x,y)$ are the points of the form $(x,y) + k(c,d)$ for $k = 0, \ldots, \bar{k}$, where $\bar{k}$ is the largest integer such that $(x,y) + \bar{k}(c,d) \in [n]^2$. Solving this last restriction for $\bar{k}$, we find that

$$x \leq n \implies \bar{k} \leq (n-x)/c \text{ and } y \leq n \implies \bar{k} \leq (n-y)/d.$$

Hence, $\bar{k} = \min\big(\lfloor (n-x)/c \rfloor, \lfloor (n-y)/d \rfloor\big)$.

In order to calculate the cardinality of a line $l_{\mathcal{B}}(x,y)$ for $(x,y) \in Z$ we pick out special subrectangles in $[n]^2$ (illustrated in Figure 2.1). First are the lower and left borders:

$$I := \{(x,y) \in [n]^2 : y \leq d\}, \quad J := \{(x,y) \in [n]^2 : x \leq c\}.$$

Define new $c \times d$ rectangles on the bottom edge, from right to left,

$$I_i := \{(x,y) \in I : n - ci < x \leq n - c(i-1)\} \quad \text{for } i = 1, \ldots, \delta,$$

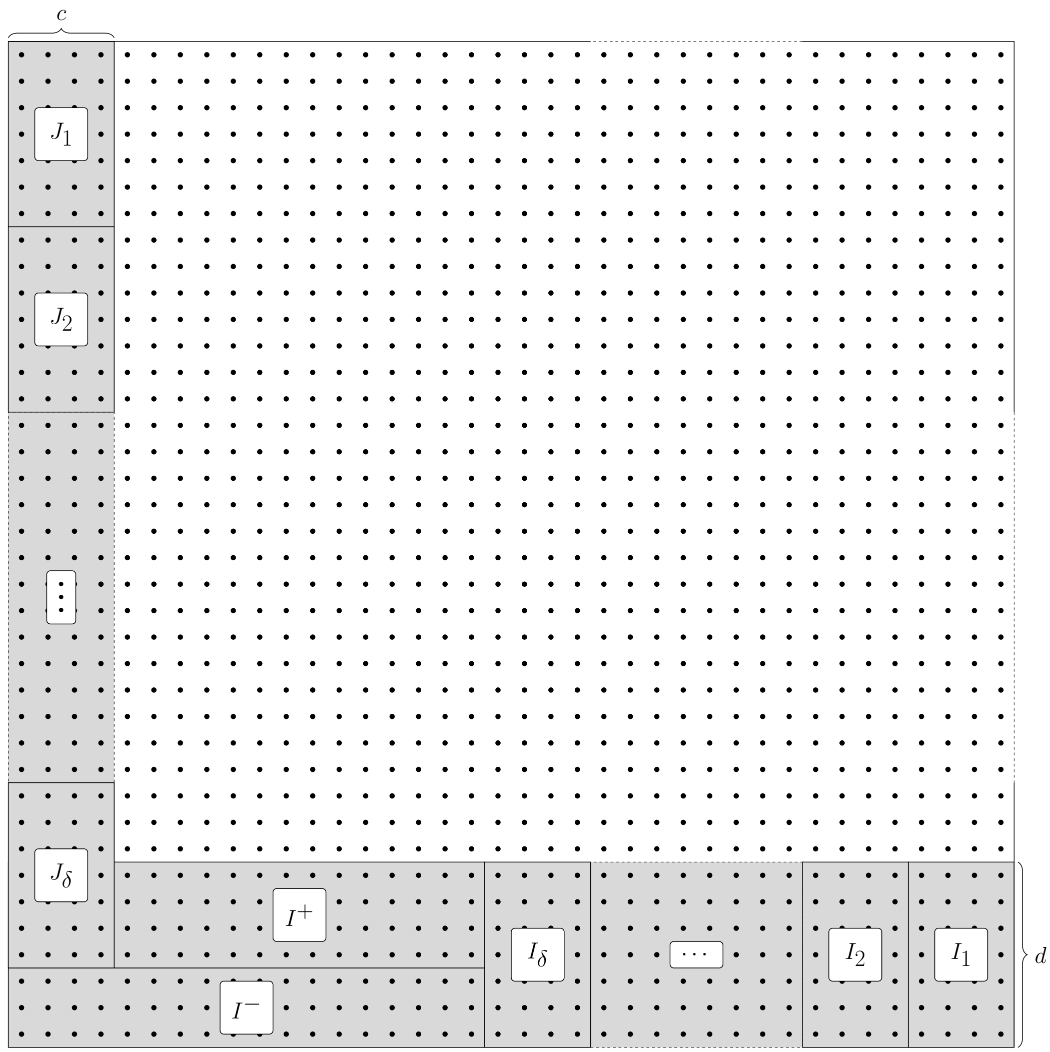

FIGURE 2.1. The division of $[n]^2$ into the shaded border region Z, with its subdividing rectangles, and the remainder of the board. (The illustration shows the case where I_δ and J_δ do not overlap.)

and on the left edge, from the top down,

$$J_j := \{(x, y) \in J : n - dj < y \leq n - d(j - 1)\} \quad \text{for } j = 1, \dots, \delta.$$

Thus, I_1 occupies the bottom right corner of $[n]^2$ and J_1 occupies the top left corner of $[n]^2$. Also, J_δ occupies the upper half of the left end of I.

Then, subdivide the remainder of Z (that is, the part of I to the left of I_δ and not in J_δ) into lower and upper halves:

$$I^- := \{(x, y) \in I : y \leq \bar{n},\ 1 \leq x \leq n - c\delta\},$$
$$I^+ := \{(x, y) \in I : y > \bar{n},\ c + 1 \leq x \leq n - c\delta\}.$$

There is a critical value of x, namely, $n - c\delta$, such that $\bar{k} = \lfloor (n-x)/c \rfloor$ if $x > n - c\delta$ and $\bar{k} = \lfloor (n-y)/d \rfloor$ if $x \le n - c\delta$. Hence, we know the size of any line by the formula

$$(2.2) \qquad |l_{\mathcal{B}}(x,y)| = \bar{k} + 1 = \begin{cases} \delta + 1, & \text{if } (x,y) \in I^-, \\ \delta, & \text{if } (x,y) \in I^+, \\ i, & \text{if } (x,y) \in I_i \text{ for } i \le \delta, \\ j, & \text{if } (x,y) \in J_j \text{ for } j \le \delta. \end{cases}$$

From Equation (2.2) we can write down the multiplicities of all line sizes in the multiset $\mathbf{L}^{d/c}(n)$ by counting the base points (x,y) in each case. We obtain the multiplicities stated in the lemma.

The rectangles I_δ and J_δ do not overlap if and only if the left end of I_δ, at $x = n - c\delta + 1$, is to the right of the right edge of J_δ, at $x = c$; that is, if and only if $n - (c+1)\delta \ge 0$; equivalently, $\delta = \lfloor n/c \rfloor$. As the width of I^+ is exactly $(n - c\delta) - c$, if there is overlap then I^+ is the overlap and has negative width, so our computation subtracts exactly the amount necessary to correct for double counting of the lines based at $(x,y) \in I_\delta \cap J_\delta$. (In this case $I^+ := \{(x,y) \in I : y > \bar{n},\ c \ge x > n - c\delta\}$.) Thus, our formula works whether or not overlap occurs. □

Now, define $\alpha^{d/c}(n)$ to be the number of ordered pairs of positions that attack each other along slope d/c. Thus,

$$\alpha^{d/c}(n) := \alpha(\mathcal{H}^{d/c}_{12}; n) = E_{(0,1)^4 \cap \mathcal{H}^{d/c}_{12}}(n+1),$$

the open Ehrhart quasipolynomial of the subpolytope $[0,1]^4 \cap \mathcal{H}^{d/c}_{12}$ of $[0,1]^4$ that satisfies the equation of attack, $(z_2 - z_1) \cdot (d, -c) = 0$, of $\mathcal{H}^{d/c}_{12}$. Counting attacking pairs of positions shows that

$$\alpha^{d/c}(n) = \sum_{l \in \mathbf{L}^{d/c}(n)} l^2.$$

The subpolytope is 3-dimensional so the degree of $\alpha^{d/c}(n)$ is 3; therefore its leading coefficient is the relative volume of $[0,1]^4 \cap \mathcal{H}^{d/c}_{12}$. Similarly, the number of ordered triples that are collinear along slope d/c is

$$\beta^{d/c}(n) := \alpha(\mathcal{W}^{d/c}_{123}; n) = E_{(0,1)^6 \cap \mathcal{W}^{d/c}_{123}}(n+1) = \sum_{l \in \mathbf{L}^{d/c}(n)} l^3.$$

whose leading coefficient is the relative volume of $[0,1]^6 \cap \mathcal{W}^{d/c}_{123}$.

The values of $\alpha^{d/c}(n)$ et al. have to be computed for each slope. Two easy examples are

$$(2.3) \qquad \alpha^{0/1}(n) = \alpha^{1/0}(n) = n^3, \qquad \alpha^{\pm 1/1}(n) = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n-1} i^2 = \frac{2n^3 + n}{3}.$$

and

$$(2.4) \qquad \beta^{0/1}(n) = \beta^{1/0}(n) = n^4, \qquad \beta^{\pm 1/1}(n) = \frac{n^4 + n^2}{2}.$$

There are general formulas.

Proposition 2.2. *For relatively prime integers $c \geq 0$ and $d > 0$ with $c \leq d$, let $\bar{n} := (n \bmod d) \in \{0, 1, \dots, d-1\}$. The number of ordered pairs of positions that attack each other along lines of slope d/c is*

$$\alpha^{d/c}(n) = \left\{ \frac{3d-c}{3d^2} n^3 + \frac{c}{3} n \right\} + \frac{\bar{n}(d-\bar{n})}{d^2} \left\{ (d-c)n - \frac{c(d-2\bar{n})}{3} \right\}. \tag{2.5}$$

The period of this quasipolynomial is d.

The number of ordered triples of positions that attack each other along a single line of slope d/c is

$$\begin{aligned} \beta^{d/c}(n) &= \left\{ \frac{2d-c}{2d^3} n^4 + \frac{c}{2d} n^2 \right\} \\ &\quad + \frac{\bar{n}(d-\bar{n})}{d^3} \left\{ 3(d-c)n^2 - (d-2c)(d-2\bar{n})n + \frac{3c\bar{n}(d-\bar{n})}{2} \right\}. \end{aligned} \tag{2.6}$$

The period of this quasipolynomial is d.

Each of these quasipolynomials has an invariant part (in the first set of braces), which is independent of $\bar{n}$, i.e., of the residue class of n, and a periodic part (in the second set of braces), which depends on $\bar{n}$. When n is a multiple of d, then $\bar{n} = 0$ and the equations reduce to the invariant part.

If the degree is e (which is 3 for $\alpha^{d/c}$ and 4 for $\beta^{d/c}$), the coefficients $\bar{\zeta}_{e-i}(\bar{n})$ of n^{e-i} of the periodic part have the alternating symmetry $\bar{\zeta}_{e-i}(d-\bar{n}) = (-1)^i \bar{\zeta}_{e-i}(\bar{n})$. For instance, in Equation (2.5) $e = 3$ and the periodic part of the coefficient of n (i.e., $i = 2$) is $\frac{\bar{n}(d-\bar{n})}{d^2}(d-c)$, which is invariant under the mapping $\bar{n} \mapsto d - \bar{n}$ (for $1 \leq \bar{n} \leq d$). That is, for any $k \in \mathbb{Z}_{>0}$ and any $\bar{n} = 1, 2, \dots, d-1$, $\bar{\zeta}_i(kd + \bar{n}) = \bar{\zeta}_i(kd + (d - \bar{n}))$ for all i.

The fact that there is no second leading term will be important in examples.

Proof. The number of attacking pairs is the sum over all lines with slope d/c of $|l_{\mathcal{B}}^{d/c}(b)|^2$. From Lemma 2.1 we can write out the total number:

$$\alpha^{d/c}(n) = 2cd \sum_{l=1}^{\delta-1} l^2 + [cd + c\bar{n} + (d-\bar{n})(n-c\delta)]\delta^2 + [\bar{n}(n-c\delta)](\delta+1)^2,$$

which simplifies to Equation (2.5) after eliminating δ via $\delta = (n - \bar{n})/d$.

If $c < d$ the period d follows from examining the coefficient of n, which equals $c/3$ only when $\bar{n} = 0$. If $c = d$, then both equal 1 and the period is $d = 1$.

The computation for attacking triples is similar. The total number of such triples is

$$\beta^{d/c}(n) = 2cd \sum_{l=1}^{\delta-1} l^3 + [cd + c\bar{n} + (d-\bar{n})(n-c\delta)]\delta^3 + [\bar{n}(n-c\delta)](\delta+1)^3,$$

which simplifies to Equation (2.6). The constant term has period exactly d. □

For the piece $\mathbb{P}$ we have the formula

$$a_{\mathbb{P}}(2; n) = \sum_{(c,d) \in \mathbf{M}} \alpha^{d/c}(n) - (|\mathbf{M}| - 1)n^2, \tag{2.7}$$

which is the sum over all moves $(c, d) \in \mathbf{M}$ of the number of placements of two labelled pieces that attack along that direction, reduced by the overcount of two pieces on the same square, which should be counted only once per square. By Equation (2.1),

$$u_{\mathbb{P}}(2;n) = \frac{1}{2!}o_{\mathbb{P}}(2;n) = \frac{1}{2}\Big[n^4 - \sum_{(c,d)\in\mathbf{M}} \alpha^{d/c}(n) + (|\mathbf{M}|-1)n^2\Big]. \tag{2.8}$$

the number of placements of two labelled pieces that do not attack each other. Then by Proposition 2.2 we have an explicit formula for $u_{\mathbb{P}}(2;n)$.

Theorem 2.3. *For $(c,d)\in\mathbf{M}$, let $\hat{c} := \min(|c|,|d|)$, $\hat{d} := \max(|c|,|d|)$, and $\bar{n} := (n \bmod \hat{d}) \in \{0,1,\ldots,\hat{d}-1\}$. On the square board,*

$$\begin{aligned} u_{\mathbb{P}}(2;n) &= \frac{1}{2!}o_{\mathbb{P}}(2;n) \\ &= \frac{1}{2}n^4 - \frac{1}{6}\sum_{r=1}^{|\mathbf{M}|}\frac{3\hat{d}_r - \hat{c}_r}{\hat{d}_r^2}n^3 + \frac{|\mathbf{M}|-1}{2}n^2 - \frac{1}{6}\sum_{r=1}^{|\mathbf{M}|}\hat{c}_r n \\ &\quad - \frac{1}{2}\sum_{r=1}^{|\mathbf{M}|}\frac{\bar{n}_r(\hat{d}_r-\bar{n}_r)(\hat{d}_r-\hat{c}_r)}{\hat{d}_r^2}n + \frac{1}{3}\sum_{r=1}^{|\mathbf{M}|}\frac{\hat{c}_r(\hat{d}_r-\bar{n}_r)(\hat{d}_r-2\bar{n}_r)\bar{n}_r}{\hat{d}_r^2}. \qquad \square \end{aligned}$$

Again, notice that the three highest terms are independent of the residue class of n.

Corollary 2.4. *The number of combinatorial types of nonattacking configuration of two pieces is the number of basic moves.*

Proof. We already proved this geometrically in Proposition I.5.6; therefore, evaluating $u_{\mathbb{P}}(2;-1)$, which is the number of types by Theorem I.5.3, checks the correctness of Theorem 2.3. We omit the computation, noting only that $\bar{n}_r = \hat{d}_r - 1$ and the result is $|\mathbf{M}|$, as it should be. $\square$

3. Coefficients of Subspace Ehrhart Functions

The more we can say about the open Ehrhart quasipolynomials $\alpha(\mathcal{U};n)$ of subspaces, the more we can infer about the configuration counting functions.

3.1. **Odd and even functions.**

The square board has a property that few other boards share (a right triangle being one that does). We remind the reader that a function $f(n)$ of an integer n is called *even* or *odd* if it satisfies $f(-n)=f(n)$ or, respectively, $f(-n)=-f(n)$. We introduce a short notation for a frequently recurring quantity. For $\mathcal{U}\in\mathscr{L}(\mathscr{A}_\mathbb{P})$, let κ be the number of pieces involved in the equations that determine $\mathcal{U}$ and let $\widetilde{\mathcal{U}}$ be the essential part of $\mathcal{U}$, i.e., the subspace of $\mathbb{R}^{2\kappa}$ that satisfies the same attack equations as $\mathcal{U}$.

Theorem 3.1 (Parity Theorem)**.** *Consider the square board with any piece $\mathbb{P}$. Let $\mathcal{U}\in\mathscr{L}(\mathscr{A}_\mathbb{P})$. The function $\alpha(\mathcal{U};n)$ is an odd function of n if $\dim\mathcal{U}$ is odd and an even function if $\dim\mathcal{U}$ is even.*

Proof. The crucial property of the square board that makes the theorem true is that the interior lattice points, $(n+1)(0,1)^2\cap\mathbb{Z}^2$, are isomorphic to all the lattice points, $(n-1)[0,1]^2\cap\mathbb{Z}^2$, by a translation. The equations of the attack hyperplanes are invariant under that translation, so the two sets of lattice points are equivalent for the inside-out Ehrhart theory of $([0,1]^{2q},\mathscr{A}_\mathbb{P})$. Therefore,

$$\alpha(\mathcal{U};n):=E_{(0,1)^{2q}\cap\widetilde{\mathcal{U}}}(n+1)=E_{[0,1]^{2q}\cap\widetilde{\mathcal{U}}}(n-1).$$

Because every $\mathcal{U}\in\mathscr{L}(\mathscr{A}_\mathbb{P})$ meets the open hypercube, $(0,1)^{2q}\cap\widetilde{\mathcal{U}}=([0,1]^{2q}\cap\widetilde{\mathcal{U}})^\circ$ and both have dimension $\dim\widetilde{\mathcal{U}}$. Therefore, $E_{(0,1)^{2q}\cap\widetilde{\mathcal{U}}}(t)=E^\circ_{[0,1]^{2q}\cap\widetilde{\mathcal{U}}}(t)$. By Ehrhart reciprocity,

$$\begin{aligned}E_{[0,1]^{2q}\cap\widetilde{\mathcal{U}}}(n-1)&=(-1)^{\dim\widetilde{\mathcal{U}}}E^\circ_{[0,1]^{2q}\cap\widetilde{\mathcal{U}}}(-(n-1))=(-1)^{\dim\widetilde{\mathcal{U}}}E^\circ_{[0,1]^{2q}\cap\widetilde{\mathcal{U}}}(-n+1)\\&=(-1)^{\dim\widetilde{\mathcal{U}}}E_{(0,1)^{2q}\cap\widetilde{\mathcal{U}}}(-n+1)=(-1)^{\dim\mathcal{U}}\alpha(\mathcal{U};-n).\end{aligned}$$

Since $\dim\mathcal{U}\equiv\dim\widetilde{\mathcal{U}}$ mod 2, that concludes the proof. □

Oddness or evenness of $\alpha(\mathcal{U};n)$ should not be confused with that of its constituent polynomials. The correct constituent properties are the following. Let $p(\mathcal{U})$ denote the period of $\alpha(\mathcal{U};n)$.

Corollary 3.2 (Constituent Parity)**.** *The constituent $\alpha_0(\mathcal{U};n)$ is an odd function of n if $\dim\mathcal{U}$ is odd and an even function if $\dim\mathcal{U}$ is even. If $p(\mathcal{U})$ is even, the middle constituent $\alpha_{p(\mathcal{U})/2}(\mathcal{U};n)$ is also odd or even, respectively. For $i\in[p(\mathcal{U})-1]$, there is the relation $\alpha_{p(\mathcal{U})-i}(\mathcal{U};n)=(-1)^{\dim\mathcal{U}}\alpha_i(\mathcal{U};-n)$.*

Thus, if the period is 2, indeed each constituent is an odd or even polynomial, but that is not necessarily so for any larger period. For example, the contribution of a hyperplane is $\alpha(\mathcal{H}^{d/c}_{12};n)=\alpha^{d/c}(n)$, given by Equation (2.5), with period d. When $d\le 2$ we get an odd polynomial in n but since $d>2$ gives a nonzero constant term the polynomial is no longer odd.

The preceding corollary can be strengthened by focussing on individual terms. Let $\mathcal{U}$ involve κ pieces and have codimension ν, and write

$$\alpha(\mathcal{U};n) := \sum_{j=0}^{2\kappa-\nu} \bar{\gamma}_j(\mathcal{U}) n^{2\kappa-\nu-j} \tag{3.1}$$

and define $p_j(\mathcal{U})$ to be the (smallest) period of $\bar{\gamma}_j(\mathcal{U})$, which may be less than the period $p(\mathcal{U})$ of $\alpha(\mathcal{U};n)$ (indeed, the latter equals $\operatorname{lcm}_j p_j(\mathcal{U})$). Thus, $\bar{\gamma}_j(\mathcal{U})$ cycles through the functions $\bar{\gamma}_{0j}(\mathcal{U}), \ldots, \bar{\gamma}_{p_j(\mathcal{U}),j}(\mathcal{U})$. If $p(\mathcal{U}) > p_j(\mathcal{U})$, then $\bar{\gamma}_{ij}(\mathcal{U})$ cycles through more than one period as $\alpha_i(\mathcal{U};n)$ goes through one of its periods. (Notational note: We employ γ_i for coefficients of an unlabelled counting function $u_{\mathbb{P}}(q;n)$ and $\bar{\gamma}_i$ for a coefficient of a labelled counting function $o_{\mathbb{P}}(q;n)$ or $\alpha(\mathcal{U};n)$.)

Corollary 3.3 (Coefficient Parity)**.** *Let $0 \le i < p_j(\mathcal{U})$. The constituents of $\bar{\gamma}_j(\mathcal{U})$ satisfy $\bar{\gamma}_{p_j(\mathcal{U})-i,\,j}(\mathcal{U}) = (-1)^{\dim \mathcal{U}-j} \bar{\gamma}_{ij}(\mathcal{U})$.*

Assume $j \not\equiv \dim \mathcal{U}$ (mod 2); then $\bar{\gamma}_{0j}(\mathcal{U}) = 0$ and (if $p_j(\mathcal{U})$ is even) $\bar{\gamma}_{p_j(\mathcal{U})/2,\,j}(\mathcal{U}) = 0$. In particular, if $p_j(\mathcal{U}) \le 2$ then $\bar{\gamma}_j(\mathcal{U}) = 0$.

Note that $\dim \mathcal{U}$ can be replaced in these formulas by $\dim \widetilde{\mathcal{U}}$ since they have the same parity.

3.2. **The second leading coefficient of a subspace Ehrhart function.**

We saw in Proposition 2.2 that $\alpha(\mathcal{U};n)$ has no second leading term (the term of $n^{2q-\operatorname{codim}\mathcal{U}-1}$) when $\mathcal{U}$ is either a hyperplane $\mathcal{H}_{ij}^{d/c}$ or a subhyperplane of the form $\mathcal{W}_{ijk}^{d/c}$. This is a general phenomenon.

Theorem 3.4. *For every subspace $\mathcal{U} \in \mathscr{L}(\mathscr{A}_{\mathbb{P}})$, the coefficient $\bar{\gamma}_1(\mathcal{U})$ of the second leading term in $\alpha(\mathcal{U};n)$ is zero.*

Proof. We apply the theorem of McMullen [4, Theorem 6] that in the (open) Ehrhart quasipolynomial $\gamma_0(\mathcal{P})n^d + \gamma_1(\mathcal{P})n^{d-1} + \cdots + \gamma_d(\mathcal{P})$ of a rational convex polytope $\mathcal{P}$ of dimension d, the coefficient $\gamma_i(\mathcal{P})$ has period that is a divisor of a quantity π_i called the i-index of $\mathcal{P}$, defined as the smallest positive integer π such that every $(d-i)$-face of $\mathcal{P}$ contains a rational point (it need not be in lowest terms) with denominator π. (This is equivalent to the face's affine span being generated by rational points with denominator π, which is McMullen's definition.) Thus, for instance, if every facet of $\mathcal{P}$ spans an affine flat that contains an integral point, the 1-index of $\mathcal{P}$ is 1, i.e., $\bar{\gamma}_1(\mathcal{P})$ is constant.

We apply McMullen's theorem to $\mathcal{P} = \mathcal{U} \cap [0,1]^{2q}$ where $\mathcal{U} \in \mathscr{L}(\mathscr{A}_{\mathbb{P}})$; that is, $\mathcal{P}$ is the part of the subspace $\mathcal{U}$ bounded by the inequalities $x_i, y_i \ge 0$ and $x_i, y_i \le 1$ for $i \in [2q]$. A face of $\mathcal{P}$ is the restriction of $\mathcal{P}$ to some subset of boundary hyperplanes; i.e., we fix some set of x_i's and y_i's to be 0 and some other set to be 1.

The equations of $\mathcal{U}$ have the form $z_j - z_i \perp (d_r, -c_r)$ for some $i < j$ and $(c_r, d_r) \in \mathbf{M}$. The equation means that $z_j - z_i$ is parallel to m_r. If $\mathcal{U}$ satisfies $z_j - z_i \parallel m_r$ for more than one basic move m_r, then $z_i = z_j$ so we can reduce q by identifying the ith and jth pieces; therefore we may assume that no pair of pieces appears in more than one equation satisfied by $\mathcal{U}$.

First, consider $i = 1$; that means we choose one x_i or y_i to be 0 or 1. Let that be x_1. Define $\Delta z := (z, z, \ldots, z) \in \mathbb{R}^{2q}$. All such points with $z \in [0,1]^2$ belong to $\mathcal{P}$; therefore in particular, each facet $\mathcal{P} \cap \{x_1 = k\}$ (where $k = 0$ or 1) contains the integral point $\Delta(k,1)$. Consequently, the index $\pi_1 = 1$. The Parity Theorem 3.1 implies then that $\bar{\gamma}_1(\mathcal{U}) = 0$. ∎

We propose a period bound for each coefficient.

Conjecture 3.5. Define $\Lambda(\mathcal{U})$ as the least common multiple of all the nonzero numbers c_r and d_r in all moves $m_r = (c_r, d_r)$ such that $\mathcal{U}$ lies in a hyperplane $\mathcal{H}_{ij}^{m_r}$ (equivalently, $\Sigma(\mathcal{U})$ contains an edge labelled m_r).

(a) The period of $\bar{\gamma}_2(\mathcal{U})$ is a divisor of $\Lambda(\mathcal{U})$.

(b) For each $i \geq 2$ there is a function $B_i(\lambda)$, independent of the piece $\mathbb{P}$, such that for every subspace $\mathcal{U}$ the period of $\bar{\gamma}_i(\mathcal{U})$ divides $B_i(\Lambda(\mathcal{U}))$.

(c) For $i = 3$ the period divides $\Lambda(\mathcal{U})$; that is, $B_3(\lambda) = \lambda$.

The conjecture is based on the evidence of hyperplanes $\mathcal{H}_{ij}^{d/c}$ and subhyperplanes $\mathcal{W}_{ijk}^{d/c}$, where the conjecture is true for every rider, and the subspaces whose counting functions $\alpha(\mathcal{U}; n)$ we compute in Parts III and IV for partial queens and nightriders, for which $\Lambda \leq 2$ so that by Theorem 3.1 $\bar{\gamma}_3(\mathcal{U})$ should be zero, as indeed it is for those pieces. The example of queens, where the period of $\bar{\gamma}_i(\mathcal{U})$ (for even i) appears to grow quickly with i, shows that in general B_i must be more complicated than a mere least common multiple.

The existence of a period bound of the form $B_i(\Lambda)$ together with an explicit formula for it would complete the theoretical solution to counting nonattacking placements of riders on a square board. It would imply a computable finite number $N_{\mathbb{P}}(q)$ such that counting the number of placements of q pieces $\mathbb{P}$ on boards of all sizes from 1 to $N_{\mathbb{P}}(q)$ would give sufficient information to determine the general formula for $u_{\mathbb{P}}(q; n)$.

4. The Form of Coefficients on the Square Board

On the square board it is possible to get fairly detailed information about the highest-order coefficients of the counting functions of nonattacking configurations.

First we list the exact contributions to $o_{\mathbb{P}}(q;n)$ of subspaces with low codimension. Much of the work of solution is in this step. For $(c,d) \in \mathbf{M}$ let $\hat{c} = \min(|c|,|d|)$ and $\hat{d} = \min(|c|,|d|)$ and define $\bar{n} := (n \bmod \hat{d}) \in \{0,1,\dots,\hat{d}-1\}$.

Lemma 4.1. *The total contribution to $o_{\mathbb{P}}(q;n)$ of the subspace of codimension 0 is n^{2q}.*

The total contribution of the subspaces of codimension 1 is $-\binom{q}{2}A_1(n)n^{2q-4}$ where

$$A_1(n) := \sum_{m\in\mathbf{M}} \alpha^m(n) = a_{10}n^3 + a_{12}n + a_{13} \tag{4.1}$$

with

$$\begin{aligned}
a_{10} &= \sum_{(c,d)\in\mathbf{M}} \frac{3\hat{d}-\hat{c}}{3\hat{d}^2},\\
a_{12} &= \sum_{(c,d)\in\mathbf{M}} \frac{\hat{c}\hat{d}^2 + 3(\hat{d}-\hat{c})\bar{n}(\hat{d}-\bar{n})}{3\hat{d}^2},\\
a_{13} &= -\sum_{(c,d)\in\mathbf{M}} \frac{\hat{c}}{3\hat{d}^2}\bar{n}(\hat{d}-\bar{n})(\hat{d}-2\bar{n}).
\end{aligned}$$

The appearance of $\bar{n}$ in a_{12} and a_{13} means that they depend on n through its residue class modulo $\hat{d}$, unlike a_{10}, which is a true constant.

Proof. The sign in $-A_1(n)$ comes from the fact that the Möbius function $\mu(\hat{0},\mathcal{H}) = -1$ for a hyperplane $\mathcal{H}$. The binomial coefficient counts the number of pairs $\{i,j\}$. The evaluation of A_1 comes from Proposition 2.2. □

Theorem 4.2 (Square-Board Coefficient Theorem)**.** *On the square board the coefficients γ_i for $i \le 2$ are independent of n. The coefficient $q!\gamma_i$ of n^{2q-i} in $o_{\mathbb{P}}(q;n)$ is a polynomial in q, of degree $2i$, which depends periodically on n. The leading coefficients are*

$$\begin{aligned}
q!\gamma_0 &= 1,\\
q!\gamma_1 &= -(q)_2\frac{a_{10}}{2},
\end{aligned}$$

and for $i \ge 1$ the coefficients are

$$\begin{aligned}
q!\gamma_i &:= (q)_{2i}\bar{\theta}_{i,2i} + (q)_{2i-1}\bar{\theta}_{i,2i-1} + \cdots + (q)_2\bar{\theta}_{i,2}\\
&= \sum_{\kappa=2}^{2i}(q)_\kappa \sum_{\nu=\lceil\kappa/2\rceil}^{\min(i,2\kappa-2)} \sum_{[\mathcal{U}_\kappa^\nu]} \mu(\hat{0},\mathcal{U}_\kappa^\nu)\frac{1}{|\operatorname{Aut}(\mathcal{U}_\kappa^\nu)|}\bar{\gamma}_{i-\nu}(\mathcal{U}_\kappa^\nu),
\end{aligned} \tag{4.2}$$

in which $\bar\theta_{i,2i} = (-a_{10}/2)^i/i!$ *and*

$$\bar\theta_{i,2} = \begin{cases} -a_{10}/2 & \text{for } i=1,\\ (|\mathbf{M}|-1)/2 & \text{for } i=2,\\ -a_{12}/2 & \text{for } i=3,\\ -a_{13}/2 & \text{for } i=4,\\ 0 & \text{for } i>4. \end{cases}$$

The period is a divisor of the least common multiple of the periods of all counting quasipolynomials $\alpha(\mathcal{U};n)$ *for* $\mathcal{U} \in \mathscr{L}(\mathscr{A}^q_{\mathbb{P}})$ *such that* $\operatorname{codim}\mathcal{U} \le i$.

Proof. The polynomiality of $q!\gamma_i$ is part of Theorem I.4.2, as is the constancy (with respect to n) of γ_0 and γ_1. The value of γ_2 is determined by subspaces of codimension 2 or less. The contribution from codimension 2 is independent of n since it is the sum of leading coefficients. By Equation (4.1), hyperplanes contribute zero. The contribution from $\mathbb{R}^{2q}$ is zero. Thus, γ_2 is independent of n.

It remains to investigate the individual coefficients $q!\gamma_i$ more closely. Because the auxiliary variable N now is simply n^2, we recalculate the formula for $o_{\mathbb{P}}(q;n)$ by rewriting Equation (I.2.1), taking account of the simple form $\alpha(\mathbb{R}^{2q};n) = n^{2q}$, substituting via Equation (3.1), and simplifying as in the proof of Theorem I.4.2, to get

$$o_{\mathbb{P}}(q;n) = n^{2q} + \sum_{\kappa=2}^{2i} (q)_\kappa \sum_{\nu=\lceil\kappa/2\rceil}^{\min(i,2\kappa-2)} \sum_{[\mathcal{U}^\nu_\kappa]} \mu(\hat 0, \mathcal{U}^\nu_\kappa) \frac{1}{|\operatorname{Aut}(\mathcal{U}^\nu_\kappa)|} \sum_{j=0}^{2\kappa-\nu} \bar\gamma_j(\mathcal{U}^\nu_\kappa) n^{2q-\nu-j}, \tag{4.3}$$

summed over subspace types $[\mathcal{U}^\nu_\kappa]$ such that $\mathcal{U}^\nu_\kappa \in \mathscr{L}(\mathscr{A}^q_{\mathbb{P}})$. So, for $i > 0$ we have

$$q!\gamma_i = \sum_{\kappa=2}^{2i} (q)_\kappa \sum_{\nu=\lceil\kappa/2\rceil}^{\min(i,2\kappa-2)} \sum_{[\mathcal{U}^\nu_\kappa]} \mu(\hat 0, \mathcal{U}^\nu_\kappa) \frac{1}{|\operatorname{Aut}(\mathcal{U}^\nu_\kappa)|} \bar\gamma_{i-\nu}(\mathcal{U}^\nu_\kappa).$$

The next step is to determine the leading term and $q!\gamma_1$. To do so we study one isomorphism type of subspace:

Type $\mathcal{U}^i_{2i}$: The subspaces $\mathcal{U}^i_{2i}$ are those isomorphic to $\mathcal{U} = \mathcal{H}^{l_1}_{12} \cap \cdots \cap \mathcal{H}^{l_i}_{2i-1,2i}$, where $l_1, \ldots, l_i \in \mathbf{M}$. The Möbius function is $\prod_j \mu(\hat 0, \mathcal{H}^{l_j}_{2j-1,2j}) = (-1)^i$. The automorphisms depend on the selection of slopes. Write $\mathbf{M} = \{m_1, m_2, \ldots, m_s\}$ where $s = |\mathbf{M}|$. Suppose k_r hyperplanes have slope m_r. Then an automorphism of $\mathcal{U}$ can reverse the subscripts in any pair and it can permute the hyperplanes with the same slope. Thus, $|\operatorname{Aut}(\mathcal{U})| = 2^i k_1! \cdots k_s!$. The value of $\alpha(\mathcal{U};n)$ is $\prod_{r=1}^s \alpha^{m_r}(n)^{k_r}$, so the total contribution of all subspaces of type $\mathcal{U}^i_{2i}$ is

$$\begin{aligned}\sum_{(k_1,\ldots,k_s)} (-1)^i \frac{1}{2^i k_1! \cdots k_s!} \prod_{r=1}^s \alpha^{m_r}(n)^{k_r} &= (-1)^i \frac{1}{2^i i!} \sum_{(k_1,\ldots,k_s)} \frac{i!}{k_1! \cdots k_s!} \prod_{r=1}^s \alpha^{m_r}(n)^{k_r} \\ &= (-1)^i \frac{1}{2^i i!} A_1(n)^i,\end{aligned}$$

the sum being taken over all s-tuples $(k_1, \ldots, k_s)$ of nonnegative integers whose total is s. Since the leading coefficient of $A_1(n)$ is a_{10}, the coefficient of $(q)_{2i}$ in Equation (4.2) is $-a_{10}/2$. Since we assumed $i > 0$, that implies the complete formula for $q!\gamma_1$.

The term of $(q)_2$ appears only when $2 \geq i/2$, i.e., $i \leq 4$. The subspaces can be $\mathcal{U}_2^1$, i.e., hyperplanes, and $\mathcal{U}_2^2$, i.e., $\mathcal{W}_{12}^{=}$ and its isomorphs.

Type $\mathcal{U}_2^1$: Every hyperplane has $\mu(\hat{0}, \mathcal{U}) = -1$ and $|\operatorname{Aut} \mathcal{U}| = 2$. Thus, the contribution of all hyperplanes is $-A_1(n)/2$. Denoting (as before) the coefficient of n^{3-j} by a_{1j}, the hyperplanes contribute $-a_{1,i-1}/2$ to $\bar{\theta}_{i2}$. That is zero when $i = 2$.

Type $\mathcal{U}_2^2$: We have $\mu(\hat{0}, \mathcal{W}_{12}^{=}) = |\mathbf{M}| - 1$ and $|\operatorname{Aut} \mathcal{W}_{12}^{=}| = 2$. Since $\alpha(\mathcal{W}_{12}^{=}; n) = n^2$, the contribution of $[\mathcal{W}_{12}^{=}]$ to $\bar{\theta}_{i2}$ is $(|\mathbf{M}|-1)\bar{\gamma}_{2-2}(\mathcal{W}_{12}^{=})/2 = (|\mathbf{M}|-1)/2$ when $i = 2$, and otherwise zero. □

If we could obtain the coefficient $\bar{\theta}_{i,2i-1}$ of $(q)_{2i-1}$ we would have, in particular, the missing coefficient $\bar{\theta}_{23}$ of a general formula for $q!\gamma_2$. There is but one difficult step in that. Since $\nu = i$,

$$\bar{\theta}_{i,2i-1} = \sum_{[\mathcal{U}]:\mathcal{U}=\mathcal{U}_{2i-1}^i} \mu(\hat{0}, \mathcal{U}) \frac{1}{|\operatorname{Aut}(\mathcal{U})|} \bar{\gamma}_0(\mathcal{U}).$$

The subspaces $\mathcal{U}_{2i-1}^i$ have the form $\big(\mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2}\big) \cap \mathcal{H}_{45}^{l_3} \cap \cdots \cap \mathcal{H}_{2i-2,2i-1}^{l_i}$, where $l_1, \ldots, l_i \in \mathbf{M}$. There are two types: $l_1 = l_2$ and $l_1 \neq l_2$. The automorphism group has order $H2^{i-2}$ where $H := |\operatorname{Aut}\big(\mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2}\big)|$. The contribution of all subspaces of either type is

$$\frac{\mu(\hat{0}, \mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2})}{H2^{i-2}} \alpha(\mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2}; n) \cdot (-1)^{i-2} A_1(n)^{i-2},$$

whose leading coefficient is

$$\frac{\mu(\hat{0}, \mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2})}{H} \Big(- \frac{a_{10}}{2} \Big)^{i-2} \bar{\gamma}_0(\mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2}).$$

So we need the value of $\bar{\gamma}_0(\mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2})$ summed over all permitted slope pairs (l_1, l_2).

Type $\mathcal{U}_{2i-1\mathrm{a}}^i$: If $l_1 = l_2$, then $\mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2} = \mathcal{W}_{123}^{l_1}$, $\mu(\hat{0}, \mathcal{W}_{123}^{l_1}) = 2$, $H = 3!$, and $\sum_{l_1} \bar{\gamma}_0(\mathcal{W}_{123}^{l_1}) = b_{10} := \sum_{(c,d)\in\mathbf{M}} (2\hat{d} - \hat{c})/2\hat{d}^3$ (from Proposition 2.2). The total contribution of this type to the coefficient is therefore

$$(-1)^i \frac{1}{3 \cdot 2^{i-2}(i-2)!} \, a_{10}^{i-2} b_{10}.$$

Type $\mathcal{U}_{2i-1\mathrm{b}}^i$: When $l_1 \neq l_2$ we have $\mu(\hat{0}, \mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2}) = 1$ and $H = 1$, but $\alpha(\mathcal{H}_{12}^{l_1} \cap \mathcal{H}_{23}^{l_2}; n)$ for arbitrary slopes l_1 and l_2 is too complicated for us. Finding just its leading coefficient would give $\bar{\theta}_{i,2i-1}$.

We propose an analog of Conjecture 3.5.

Conjecture 4.3. Define Λ as the least common multiple of all the nonzero numbers c_r and d_r in all moves $m_r = (c_r, d_r)$. (It equals $\Lambda(\mathcal{W}_{ij\ldots}^{=})$.)

(a) The period of γ_2 is Λ.

(b) For each $i \geq 2$ there is a function $B_i(\lambda)$, independent of the piece $\mathbb{P}$, such that the period of γ_i equals $B_i(\Lambda)$. (The function is the same as in Conjecture 3.5.)

(c) For $i = 3$ the period is Λ.

5. One-Move Riders

Theorem 4.2 is best possible for pieces in general. Even for a piece with only one attacking move, the coefficient γ_3 may vary with n with a large period. We prove that here (without appealing to the general theory). This leads us to propose that periodic variability of higher quasipolynomial coefficients occurs, not due to the number of attacking moves, but because of their slopes.

Consider a piece $\mathbb{P}$ with move set $\mathbf{M} = \{(c,d)\}$, where c and d are relatively prime integers such that $0 < c \le d$. By symmetry, this covers all types of move except $(1,0)$ and $(0,1)$, which are elementary.

Proposition 5.1. *For a piece $\mathbb{P}$ with move set $\mathbf{M} = \{(c,d)\}$ where $0 < c \le d$,*

$$
\begin{aligned}
u_{\mathbb{P}}(1;n) &= n^2,\\
u_{\mathbb{P}}(2;n) &= \frac{1}{2}\Big\{ n^4 + \frac{c-3d}{3d^2}n^3 - \frac{c}{3}n \Big\}\\
&\quad + \frac{\bar n(d-\bar n)}{6d^2}\Big\{ [3(c-d)]n + c(d-2\bar n) \Big\},\\
u_{\mathbb{P}}(3;n) &= \frac{1}{6}\Big\{ n^6 + \frac{c-3d}{d^2}n^5 - \frac{c-2d}{d^3}n^4 - cn^3 + \frac{c}{d}n^2 \Big\}\\
&\quad + \frac{\bar n(d-\bar n)}{6d^3}\Big\{ \big[3d(c-d)\big]n^3 + \big[6d + cd^2 - 2cd\bar n - 6c\big]n^2\\
&\qquad\qquad + \big[2(d-2c)(d-2\bar n)\big]n + 3c(d-\bar n)\bar n \Big\},\\
u_{\mathbb{P}}(4;n) &= \frac{1}{24}\Big\{ n^8 + \frac{2(c-3d)}{d^2}n^7 + \frac{c^2-18cd+33d^2}{3d^4}n^6 + \frac{18c-30d-10cd^4}{5d^4}n^5\\
&\qquad + \frac{18cd-2c^2}{3d^2}n^4 - \frac{4c}{d^2}n^3 + \frac{c^2}{3}n^2 + \frac{2c}{5}n \Big\}\\
&\quad + \frac{\bar n(d-\bar n)}{360d^4}\Big\{ \big[90d^2(c-d)\big]n^5 + \big[30(c^2+15d^2-16cd+cd^3-2cd^2\bar n)\big]n^4\\
&\qquad + 10\big[6d(-9+2d(d-2\bar n)) + c^2(d-2\bar n) + 27c(2-d^2+2d\bar n)\big]n^3\\
&\qquad + 15\big[2d(-12d+c(18-cd+d^2)) + 3(-24c+(16+c^2)d+2cd^2+d^3)\bar n\\
&\qquad\qquad - 3(c+d)^2\bar n^2\big]n^2\\
&\qquad + 10\big[9cd^2-9d^3-c^2d^3-(27d^2-81cd+5c^2d^2-3cd^3)\bar n\\
&\qquad\qquad + 9(3d+c(cd-d^2-9))\bar n^2 + 6c(d-c)\bar n^3\big]n\\
&\qquad + c(d-2\bar n)\big[d^2(-6+5c\bar n) - 3d\bar n(36+5c\bar n) + 2\bar n^2(54+5c\bar n)\big] \Big\}.
\end{aligned}
$$

Proof. The formula for $u_{\mathbb{P}}(1;n)$ is trivial. Theorem 2.3 implies the value of $u_{\mathbb{P}}(2;n)$; a combinatorial count similar to that for $u_{\mathbb{P}}(3;n)$ and $u_{\mathbb{P}}(4;n)$ gives the same result.

Direct combinatorial arguments for $q = 3$ and 4 give

$$u_{\mathbb{P}}(3;n) = \binom{n^2}{3} - \sum_{l\in\mathbf{L}(n)} \binom{l}{3} - \sum_{l\in\mathbf{L}(n)} \binom{l}{2}\big[n^2 - l\big] \qquad \text{and}$$

$$u_{\mathbb{P}}(4;n) = \binom{n^2}{4} - \sum_{l\in\mathbf{L}(n)} \binom{l}{4} - \sum_{l\in\mathbf{L}(n)} \binom{l}{3}\big[n^2 - l\big] - \sum_{\{l,l'\}\subseteq\mathbf{L}(n)} \binom{l}{2}\binom{l'}{2}$$
$$- \sum_{l\in\mathbf{L}(n)} \binom{l}{2}\left[\binom{n^2-l}{2} - \sum_{l'\in\mathbf{L}(n)} \binom{l'}{2} + \binom{l}{2}\right],$$

where $\mathbf{L}(n) := \mathbf{L}^{d/c}(n)$. For instance, $u_{\mathbb{P}}(4;n)$ is the number of placements of four non-attacking pieces, which we count by placing four pieces on any of the n^2 positions on the board and removing those where at least two pieces attack. We must remove the cases where four pieces are in the same line $l^{d/c}(b)$, those where three pieces are in the same line and the fourth is in another line, those where two pieces are in the same line $l^{d/c}(b)$ and the remaining two are both in another line $l^{d/c}(b')$, and last, those where two pieces are attacking and the remaining two pieces attack none of the others.

The reasoning for $u_{\mathbb{P}}(3;n)$ is simpler so we merely show the steps involved in the development of the formulas:

$$\begin{aligned}
u_{\mathbb{P}}(3;n) &= \binom{n^2}{3} - \sum_{l\in\mathbf{L}(n)} \binom{l}{3} - \sum_{l\in\mathbf{L}(n)} \binom{l}{2}\big[n^2 - l\big] \\
&= \binom{n^2}{3} + 2\sum_{l\in\mathbf{L}(n)} \binom{l}{3} - (n^2-2)\sum_{l\in\mathbf{L}(n)} \binom{l}{2} \\
&= \binom{n^2}{3} + 2\Big\{2cd\binom{\frac{n-\bar n}{d}}{4} + \Big[(d-\bar n)\big(n - c\frac{n-\bar n}{d}\big) + c(\bar n + d)\Big]\binom{\frac{n-\bar n}{d}}{3} \\
&\qquad\qquad + \Big[\bar n\big(n - c\frac{n-\bar n}{d}\big)\Big]\binom{\frac{n-\bar n}{d}+1}{3}\Big\} \\
&\quad - (n^2-2)\Big\{2cd\binom{\frac{n-\bar n}{d}}{3} + \Big[(d-\bar n)\big(n - c\frac{n-\bar n}{d}\big) + c(\bar n + d)\Big]\binom{\frac{n-\bar n}{d}}{2} \\
&\qquad\qquad + \Big[\bar n\big(n - c\frac{n-\bar n}{d}\big)\Big]\binom{\frac{n-\bar n}{d}+1}{2}\Big\},
\end{aligned}$$

which when expanded (we used Mathematica) gives the constant and periodic parts stated in the proposition. □

The equation for $u_{\mathbb{P}}(3;n)$ agrees with the formulas for partial queens $\mathbb{Q}^{10}$ and $\mathbb{Q}^{01}$ in Part III. It would be instructive to find $u_{\mathbb{P}}(q;n)$ in general, but this task seems difficult.

The example with $(c,d) = (1,2)$ simplifies nicely when $q = 2, 3, 4$.

Corollary 5.2. *For a piece $\mathbb{P}$ with move set $\mathbf{M} = \{(1,2)\}$, the following formulas hold.*

$$u_{\mathbb{P}}(2;n) = \left\{\frac{n^4}{2} - \frac{5n^3}{24} - \frac{11n}{48}\right\} + (-1)^n\frac{n}{16},$$

$$u_{\mathbb{P}}(3;n) = \left\{\frac{n^6}{6} - \frac{5n^5}{24} + \frac{n^4}{16} - \frac{11n^3}{48} + \frac{7n^2}{48} + \frac{1}{32}\right\} + (-1)^n\left\{\frac{n^3}{16} - \frac{n^2}{16} - \frac{1}{32}\right\}$$

$$u_{\mathbb{P}}(4;n) = \left\{\frac{n^8}{24} - \frac{5n^7}{48} + \frac{97n^6}{1152} - \frac{131n^5}{960} + \frac{223n^4}{1152} - \frac{17n^3}{192} + \frac{137n^2}{2304} - \frac{73n}{1920}\right\} + (-1)^n \left\{\frac{n^5}{32} - \frac{29n^4}{384} + \frac{3n^3}{64} - \frac{35n^2}{768} + \frac{7n}{128}\right\}.$$

Proof. The value of $u_{\mathbb{P}}(2;n)$ is simplified from Proposition 5.1. That of $u_{\mathbb{P}}(3;n)$ is obtained by simplifying the formula

$$u_{\mathbb{P}}(3;n) = \begin{cases} \binom{n^2}{3} + 8\binom{n/2}{4} + 2(n+2)\binom{n/2}{3} \\ \qquad -(n^2-2)\left[4\binom{n/2}{3} + (n+2)\binom{n/2}{2}\right] & \text{for } n \text{ even,} \\ \binom{n^2}{3} + 8\binom{(n+1)/2}{4} + (n-1)\binom{(n-1)/2}{3} + (n+1)\binom{(n+1)/2}{3} \\ \qquad -(n^2-2)\left[4\binom{(n+1)/2}{3} + \frac{n-1}{2}\binom{(n-1)/2}{2} + \frac{n+1}{2}\binom{(n+1)/2}{2}\right] & \text{for } n \text{ odd,} \end{cases}$$

also from Proposition 5.1. The simplification of $u_{\mathbb{P}}(4;n)$ is similar. □

In each formula the fourth coefficient has period 2. This and Lemma 2.1 suggest a generalization.

Conjecture 5.3. For a one-move rider with basic move (c,d), the period of γ_3 in $u_{\mathbb{P}}(q;n)$ is $\max(|c|,|d|)$.

This and the bishop, queen, and nightrider suggest a greater generalization.

Conjecture 5.4. For any rider, the period of γ_3 in $u_{\mathbb{P}}(q;n)$ is Λ, the least common multiple of the numbers $\max(|c|,|d|)$ for all basic moves $(c,d) \in \mathbf{M}$.

The number of combinatorial types is obviously 1 (as stated in Theorem I.5.8). This implies a check on any formula for $u_{\mathbb{P}}(q;n)$, since $u_{\mathbb{P}}(q;-1)$ must equal 1. Applying the check to $u_{\mathbb{P}}(2;n)$, $u_{\mathbb{P}}(3;n)$, and $u_{\mathbb{P}}(4;n)$ in Proposition 5.1, realizing that $\bar{n} = d-1$, does give $u_{\mathbb{P}}(q;-1) = 1$ for $q = 2,3,4$.

6. A Formula for the n-Queens Problem

Theorem 4.2 covers any number of pieces on any size board. By setting $q = n$ we obtain what can be regarded as the first closed-form formula (according to [1]) for the n-Queens Problem, which is the case in which $\mathbb{P}$ is the queen in the following result. Let $\mathscr{A}_{\mathbb{P}}^{\infty}$ be the arrangement in the countably-infinite-dimensional vector space $\mathbb{R}^{\infty}$ of all move hyperplanes $\mathcal{H}_{ij}^{d/c}$, $\{i, j\} \subset \mathbb{Z}_{>0}$.

Theorem 6.1. *The number of ways to place n unlabelled copies of a rider piece $\mathbb{P}$ on an $n \times n$ board so that none attacks another is*

$$u_{\mathbb{P}}(n; n) = \frac{1}{n!} \sum_{i=1}^{2n} n^{2n-i} \sum_{\kappa=2}^{2i} (n)_{\kappa} \sum_{\nu=\lceil \kappa/2 \rceil}^{\min(i, 2\kappa-2)} \sum_{[\mathcal{U}_{\kappa}^{\nu}] : \mathcal{U}_{\kappa}^{\nu} \in \mathscr{L}(\mathscr{A}_{\mathbb{P}}^{\infty})} \mu(\hat{0}, \mathcal{U}_{\kappa}^{\nu}) \frac{1}{|\operatorname{Aut}(\mathcal{U}_{\kappa}^{\nu})|} \bar{\gamma}_{i-\nu}(\mathcal{U}_{\kappa}^{\nu}).$$

,

This formula is very complicated and potentially infinite (potentially rather than actually, because for each value of n the number of nonzero terms is finite) but it is explicitly computable. We have not tried to compare its complexity with that of other methods of counting nonattacking placements.

7. Questions, Extensions

Work on nonattacking chess placements raises many questions, several of which have general interest. Besides Conjectures 3.5, 4.3, 5.3, and 5.4, and others to appear in later parts, we propose the following directions for research.

7.1. **Detailed improvements.**

These problems concern significant loose ends we left in basic counting questions.

(a) Generalize Proposition 2.2 by finding a formula for the number of ways to place q mutually attacking pieces on the same slope line. The starting point is that the number of such placements in a line of length l is l^q, which would be summed. A consequence by inclusion–exclusion will be a complete solution for one-move pieces, which in turn may suggest general results about periods.

(b) Extend the formulas for $u_{\mathbb{P}}(q;n)$ for $q \leq 4$ for a general one-move rider $\mathbb{P}$ (Section 5) to larger numbers of pieces. This should give more indications of the behavior of periods.

(c) Evaluate the coefficient of $(q)_3$ in $q!\gamma_2$ for an arbitrary rider in Theorem 4.2 to get a complete formula for γ_2.

7.2. **Recurrences and their lengths.**

Kotěšovec obtained empirical formulas for $u_{\mathbb{P}}(q;n)$ for relatively large numbers q of various pieces (queen, bishop, nightrider) by computing the values for $n = 1, 2, \ldots, N$ where N is fairly large, and looking for a heuristic recurrence relation. He derives a generating function from that recurrence, then uses the generating function to get a quasipolynomial formula. Since the recurrence is heuristic, the formula is unproved. To prove his formula, if the period is p, he has to compute up to about $N = 2qp$—and worse, p is unknown—but if the recurrence is much shorter than p, he will find a recurrence from a much smaller value of N. That seems always to be the case (if p is large). In other words, there seems to be a recurrence for $u_{\mathbb{P}}(q;n)$ that is much shorter than p. Can our method explain this—or even prove it?

The length of the recurrence is the degree of the denominator of the generating function of $u_{\mathbb{P}}(n)$ when it is reduced to lowest terms. The explanation of a (relatively) short recurrence is that the generating function, which has the standard form $f(x)/(1-x^p)^{2q}$ where $f(x)$ is a polynomial and p is the period, is not in lowest terms. Thus, there seems to be a systematic common factor of the numerator and denominator when expressed in standard form. (One instance is Kotěšovec's conjecture about the denominator for q queens [3, 2nd ed., p. 14; 6th ed., p. 22].) Essentially nothing is known about the presumed common factor, starting with why it exists. This seems the most important research problem in the subject.

Dictionary of Notation

This dictionary refers to the initial description of the notation in this article, where applicable. The reader may wish to refer to the dictionary of notation from Part I as well (some cross-references are supplied here).

a_{1i} – coeffs of $A_1(n)$ (p. 12)
$a_{\mathbb{P}}(2;n)$ – # attacking configurations (p. 4)
b – y-intercept of $l^{d/c}(b)$ (p. 4)
$(c,d),(c_r,d_r)$ – coords of basic move (p. 4)
d/c – slope of a line (p. 4)
$(\hat{c},\hat{d})$ – $(\min,\max)$ of c,d (p. 8)
l – index for $\mathbf{L}^{d/c}(n)$; i.e., line size (p. 4)
$l^{d/c}(b)$ – line of slope d/c, y-int b (p. 4)
$l^{d/c}_{\mathcal{B}}(b) = l^{d/c}(b) \cap [n]^2$ (p. 4)
$m_r = (c_r,d_r)$ – basic move (p. 4)
n – size of square board (p. 2)
$n+1$ – dilation factor for board (p. 2)
$[n] = \{1,\dots,n\}$ (p. 2)
$[n]^2$ – square board (p. 2)
$\bar{n}$ – $n \bmod \hat{d}$ (p. 4)
$o_{\mathbb{P}}(q;n)$ – # nonattacking lab configs (p. 2)
p – period of counting quasipoly (p. 3)
$p(\mathcal{U})$ – period of quasipoly $\alpha(\mathcal{U};n)$ (p. 3)
$p_j(\mathcal{U})$ – period of $\bar{\gamma}_j(\mathcal{U})$ (p. 3)
q – # pieces on a board (p. 2)
r – move index
$u_{\mathbb{P}}(q;n)$ – # nonattack unlab configs (p. 2)
$z=(x,y), z_i=(x_i,y_i)$ – piece position (p. 4)

$\alpha(\mathcal{U};n)$ – quasipoly for subspace (p. 9)
$\alpha_i(\mathcal{U};n)$ – constituent of $\alpha(\mathcal{U})$ (p. 9)
$\alpha^{d/c}(n)$ – # 2-piece collinear attacks (p. 6)
$\beta^{d/c}(n)$ – # 3-piece collinear attacks (p. 6)
γ_i – coeff in unlab counting fcn $u_{\mathbb{P}}$ (p. 2)
$\bar{\gamma}_j(\mathcal{U})$ – coeff in counting fcn $\alpha(\mathcal{U};n)$ (p. 10)
δ_{ij} – Kronecker delta (p. 3)
$\delta = \lfloor n/d \rfloor$ (p. 4)
$\bar{\zeta}_i$ – coefficient in periodic part (p. 7)
$\bar{\theta}_{i,\kappa}$ – coefficient of $(q)_\kappa$ in $q!\gamma_i$ (p. 12)
κ – # pieces involved in subspace (p. 9)
μ – Möbius function of $\mathscr{L}(\mathscr{A})$ (p. 12)
ν – codim of subspace (p. 9)
π_i – i-index of $\mathcal{P}$ (p. 10)

$A_1(n)$ – # attacking ordered pairs (p. 12)
$B_i(\mathbf{M})$ – period bound on quasipoly coeff (p. 11)
D – denom of (inside-out) polytope (pp. I.6, 2)
$E_{\mathcal{P}}$ – Ehrhart quasipoly (p. I.6)
$E^\circ_{\mathcal{P}}$ – open Ehrhart quasipoly (p. I.6)
$E^\circ_{\mathcal{P},\mathscr{A}}$ – open Ehrhart of inside-out poly (p. I.6)
H – size of automorphism group (p. 14)
$I,\ J,\ I_i,\ J_j$ – subsets of Z (p. 4)
N – aux variable $= n^2$ in Part II (pp. I.14, 13)
Z – lower left border of $[n]^2$ (p. 4)

$\mathbf{L}^{d/c}(n)$ – multiset of line sizes (p. 4)
$\mathbf{M}$ – set of basic moves (p. 2)

$\mathscr{A}_{\mathbb{P}}$ – move arr of piece $\mathbb{P}$ (p. I.9)
$\mathcal{B},\mathcal{B}^\circ$ – closed, open board polygon (p. 2)
$\mathcal{H}^{d/c}_{ij},\mathcal{H}^m_{ij}$ – hyperpl for move $m=(c,d)$ (p. 14)
$\mathscr{L}$ – intersection semilattice (p. I.6)
$\mathcal{P}$, $\mathcal{P}^\circ$ – polytope, open polytope (p. 2)
$(\mathcal{P},\mathscr{A}_{\mathbb{P}})$ – inside-out polytope (p. I.6)
$\mathcal{U}$ – subspace in intersection semilatt (p. 9)
$[\mathcal{U}]$ – subspace type (p. I.11)
$\widetilde{\mathcal{U}}$ – essential part of $\mathcal{U}$ (p. 9)
$\mathcal{W}^{d/c}_{ij\ldots}$ – subspace of slope relation (p. I.10)
$\mathcal{W}^{=}_{ij\ldots}$ – subspace of equal position (p. I.10)

$\mathbb{Q}$ – rational numbers
$\mathbb{R}$ – real numbers
$\mathbb{Z}$ – integers

$\mathbb{P}$ – piece (p. 2)

$\Lambda(\mathcal{U})$, Λ – lcm of moves (pp. 11, 14)
$\mathbf{\Sigma}(\mathcal{U})$ – slope graph (p. I.11)

$\mathrm{Aut}(\mathcal{U})$ – subspace automorphisms
$\mathrm{codim}(\mathcal{U})$ – subspace codimension
$\dim(\mathcal{U})$ – subspace dimension

References

[1] Jordan Bell and Brett Stevens, A survey of known results and research areas for n-queens. *Discrete Math.* **309** (2009), no. 1, 1–31.

[2] Seth Chaiken, Christopher R. H. Hanusa, and Thomas Zaslavsky, A q-queens problem. I. General theory. Submitted. III. Partial queens. IV. Queens, bishops, and nightriders. V. The bishops' period. In preparation.

[3] Václav Kotěšovec, *Non-attacking chess pieces (chess and mathematics)* [*Šach a matematika - počty rozmístění neohrožujících se kamenů*]. (In mixed Czech and English.) [Self-published online book], Apr. 2010; 2nd ed. Jun. 2010; 3rd ed. Jan., 2011; 4th ed. June, 2011; 5th ed. Jan., 2012; 6th ed. Feb., 2013, 795 pp.
URL `http://web.telecom.cz/vaclav.kotesovec/math.htm`

[4] P. McMullen, Lattice invariant valuations on rational polytopes. *Arch. Math. (Basel)* **31** (1978/1979), no. 5, 509–516. MR 526617 (80d:52011). Zbl 387.52007.

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