Turning Mathematics into Jewelry

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Recent developments in 3D modeling make it possible to bring mathematical concepts to a wide audience. I have used various 3D design techniques in Mathematica to create precise and intricate jewelry models based on mathematical ideas. Some of the models are 3D printed in nylon and dyed in bright colors while others are 3D printed in metals such as brass or silver through a lost-wax casting method.

Here I will share mathematical ideas behind some of the pieces of jewelry I have created for Hanusa Design along with artistic decisions made to balance the ideal mathematical concept with the final physical form.

Koch Tetrahedron

Start with an equilateral triangle. Replace the middle third of each of the edges by a bump that would form an equilateral triangle with the removed segment. Repeat this same process on the new shape, replacing the middle third of each line segment by an equilateral bump. Iterating this process indefinitely results in a well-known fractal called The *Koch snowflake*. Figure 1 demonstrates the first three iterations of the snowflake: the newly added edges at each step have been colored.
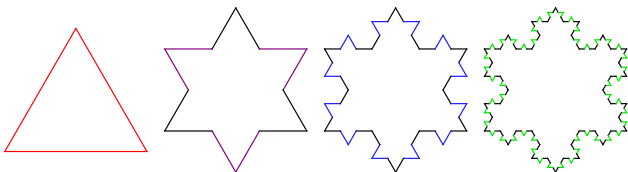

Figure 1. Building the Koch snowflake.

By construction, the area of the limiting shape is bounded; however, the limiting shape also has arbitrarily large perimeter! (You can see this because the total perimeter is multiplied by $4/3$ in every iteration.) I used the third iteration of the Koch snowflake to design a pendant (figure 2).

Figure 2. A pendant based on the Koch snowflake.

The *Koch tetrahedron* generalizes the Koch snowflake to the third dimension. In this case, begin with a tetrahedron and iteratively apply the following transformation to every triangular face. Divide the triangular face into four congruent equilateral triangles. The center triangle of this dissection (one quarter of the area of the face) is replaced by a bump that forms a regular tetrahedron. Now, the volume of this object is bounded while its surface area grows without bound. (The multiplicative factor is 6/4 instead of 4/3.) Figure 3 demonstrates the first three iterations of the construction where the newly added faces at each step have been colored.
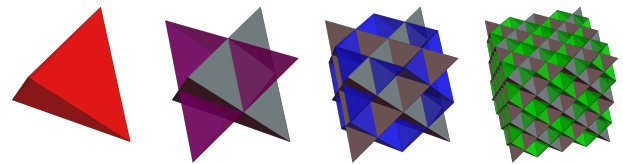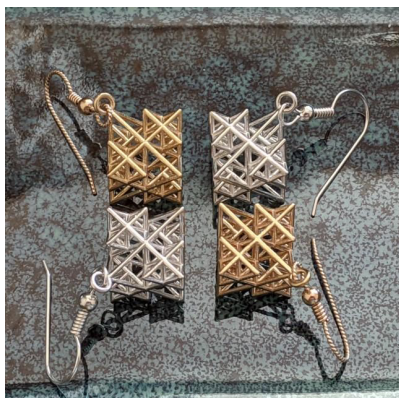

Figure 3. Building the Koch tetrahedron.

To turn this construction into jewelry, I partially performed two iterations in order to better show the smallest tetrahedra. A fascinating property of the limiting fractal shape of the Koch Tetrahedron

is that it nestles perfectly inside a cube; this is visible in the pairs of earrings shown in figure 4.

Figure 4. Koch tetrahedron earrings.

Apollonian circle packing
A *circle packing* of a region is an arrangement of a collection of non-overlapping circles in the region where none of the circles can be enlarged without overlapping another circle or the boundary of the region. One mathematically rich circle packing is known as an *Apollonian gasket*. Start with three mutually tangent circles, each tangent to a fourth circular boundary, as shown on the left in figure 5. Iteratively add the inscribed circle for every triple of mutually tangent circles. The first four circles added appear in the middle image in figure 5. As this process continues, the space between the placed circles fills in with smaller and smaller circles, as shown on the right in figure 5.
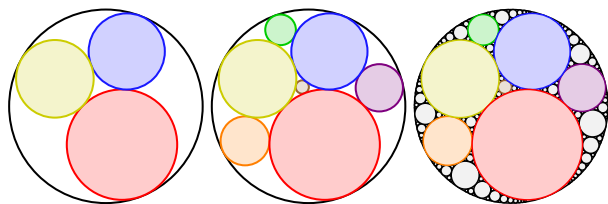
Figure 5. Iteratively building an Apollonian gasket.

If the boundary circle is the unit circle and the curvature (defined as the reciprocal of the radius, $1/r$) of each initial circle is an integer, then every circle in this construction also has integral curvature! (See Graham et al., "Apollonian circle packings: number theory," *J. Number Theory*, **100**(1) 1–45 [2003] for more details and beautiful visualizations.) A blog post by Brent Yorgey inspired me to program this construction, which led to two distinctive pairs of earrings and a necklace. I especially enjoyed the result of removing the outer circle in my mismatched pair

of *Bubbly Apollonian Earrings* (shown in purple nylon in figure 6).

Figure 6. Bubbly Apollonian Earrings.

Cobweb Plots
Suppose you take the unit interval $X = [0,1]$ and a function $f : X \to X$. Iterating the function to build the infinite sequence
$$(x, f(x), f^2(x), f^3(x),\ldots),$$
where $f^n(x) = (f \circ f^{n-1})(x)$, creates a discrete dynamical system for any initial value $x \in X$. A *cobweb plot* of this system is a two-dimensional visualization of the dynamics of this one-dimensional system, created by a sequence of line segments connecting the points $(f^i(x), f^i(x))$, $(f^i(x), f^{i+1}(x))$, and $(f^{i+1}(x), f^{i+1}(x))$ for all $i \geq 0$. The construction of the cobweb plot manifests itself as a path that bounces back and forth between the graph of the function $y = f(x)$ and the graph of the diagonal line $y = x$. Figure 7 is an example when *f* is a cubic function and the initial *x*-value is 0.8995.

i	$f^i(x)$
0	0.8995
1	0.125
2	0.926
3	0.196
4	0.998
5	0.488
6	0.530
7	0.422
8	0.698
9	0.067
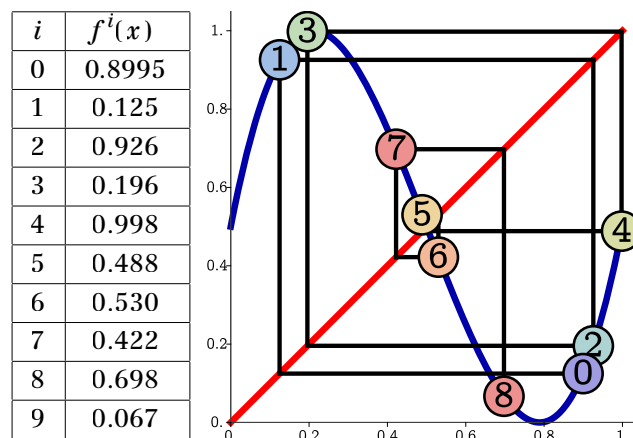
Figure 7. A cubic function (in blue), the line y = x (in red) and the cobweb plot on 8 iterations of the function starting at 0.8995.

Katherine Moore introduced me to this construction in a talk about her joint work with Sergi Elizalde, in which they considered situations where the sequence is periodic ("Characterizations and enumerations of patterns of signed shifts," *Discrete Appl. Math.*, **277** 92–114 [2020]).

The sequences can be ensured to be periodic for certain piecewise-linear functions. In such a case, the cobweb plot is a closed loop. To create the *Cobweb Earrings* shown in figure 9, I chose two different starting values of x for the three-piece function pictured on the left of figure 8. To create the pendant for the *Cobweb Necklace* in figure 9, I used the four-piece function on the right in figure 8. To add a little artistic flair, I added a minor depth change in the z-coordinate at each turn. The end result is rather stunning.
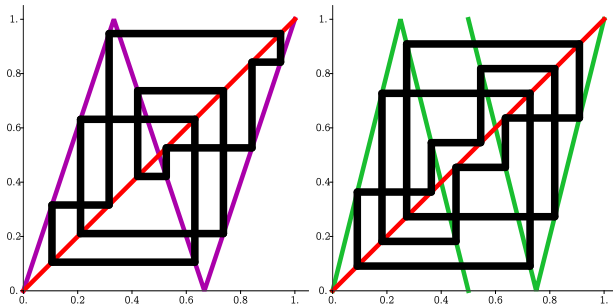
**Figure 8.** Two piecewise graphs used to create cobwebs for the jewelry in figure 9.

Figure 9. Cobweb Earrings and Cobweb Necklace.
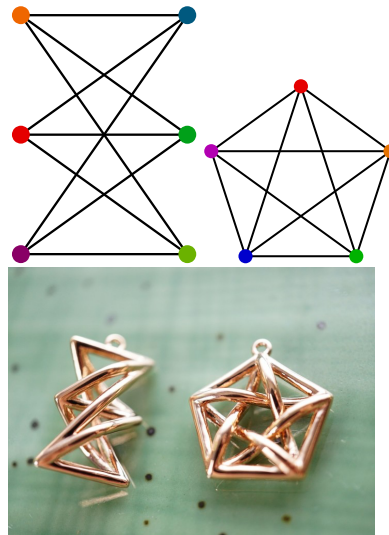
Forbidden Subgraph Earrings
In graph theory, a *graph* consists of an abstract collection of nodes and the edges that connect them. We call a graph *planar* if it can be drawn as a collection of points and connecting arcs in the plane without any of the arcs crossing each other and *non-planar* otherwise. For instance, the graph drawn on the left of figure 10 is planar even though it has overlapping edges, because it can be drawn as in the middle of the figure with no edges overlapping. However, there is no way to draw the

graph on the right of figure 10 without overlapping edges. (This third graph is called the Petersen graph, and is notorious for being a counterexample to numerous conjectures.)
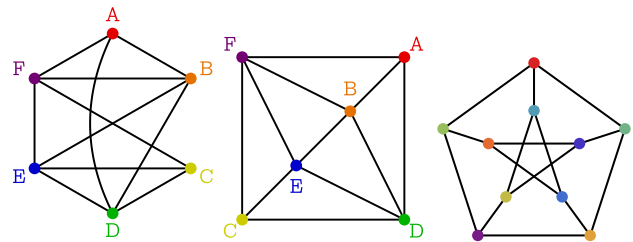
Figure 10. The two graphs on the left are the same, and thus planar, while the Petersen graph, on the right, is non-planar.

There are many ways to draw any one graph, so it would be nice to have a condition describing when an abstract graph is planar (even a large one). Luckily for us, *Kuratowski's theorem* (also known as *Pontryagin's theorem*) tells us that there are two forbidden configurations of nodes and edges in planar graphs. The two configurations are the complete bipartite graph on six vertices and the complete graph on five vertices (pictured in figure 11). A graph is non-planar if and only if it contains one of these configurations. I created nonintersecting three-dimensional embeddings of these two graphs as a pair of *Forbidden Subgraph Earrings*. (Remember, they can't be drawn as nonintersecting two-dimensional embeddings!) When you wear these earrings, you're wearing a theorem.

Figure 11. The complete bipartite graph on six vertices (left), the complete graph on five vertices (middle), and the Forbidden Subgraph Earrings they inspired (right).
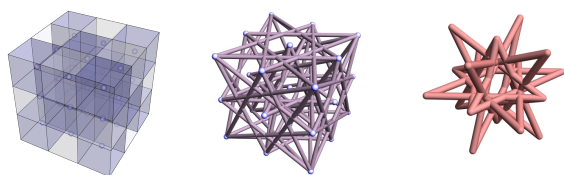
Knight's Tour Earrings

The chessboard serves as a playground for various recreational mathematical ideas. For instance, a knight moves two squares horizontally and one square vertically (or vice versa). This type of move leads to an interesting question: Is it possible for a knight to move around an $m \times n$ chessboard in such a way that it will visit every square once before returning to its initial position? This is called a (closed) knight's tour of the chessboard. Figure 13 shows one of the approximately 1.32×10^{13} knight's tours of the 8×8 chessboard.
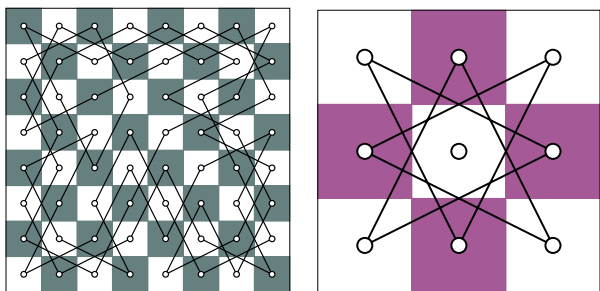
Figure 12. A knight's tour on the standard chessboard (left), and a chessboard with no knight's tour (right).

On the other hand, it is not possible for there to exist a knight's tour of the 3×3 chessboard in figure 13. First, a knight placed at the center square has no place to move. Second, the sequence of moves that a knight visits on a tour must always alternate between black and white squares. This means that if both *m* and *n* are odd, the numbers of black and white squares are different, so a knight's tour would be impossible.

If we remove the central square in the 3×3 chessboard, figure 13 shows a knight's tour on the remaining board. I took this idea to the next dimension to create jewelry. For a 3D chessboard—made of cubes instead of squares—I decided that a knight would move two cubes in one direction and one cube in an orthogonal direction. Figure 14 contains a visualization of the set of all possible such 3D knight moves on a $3 \times 3 \times 3$ chessboard. If we remove all moves that visit a corner, the remaining edges have a three-dimensional star shape; these form the basis for my *Starry Knight Earrings*, pictured in figure 15.


Figure 13. A 3D chessboard, all possible 3D knight moves, and moves that don't visit a corner.

Figure 14. Starry Knight Earrings.

If we instead try to form a knight's tour on the $3 \times 3 \times 3$ chessboard, once again the center position is inaccessible from the others. However, upon removing that position from consideration, the numbers of black and white cubes are off by two. Strategically removing two additional cubes of the same color makes a knight's tour possible; I used Mathematica to choose two random knight's tours of the resulting graph. Can you find where the missing cubes are? As an additional artistic choice, these *Knight's Tour Earrings* are a mismatched pair colored black and white to highlight the connection to chess.
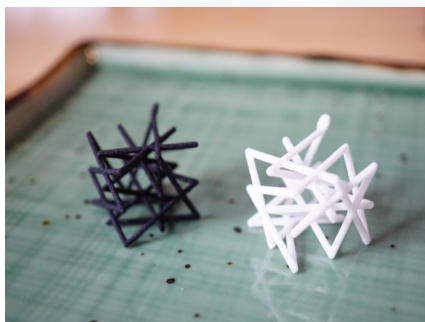
Figure 15. Knight's Tour Earrings.

Concluding Thoughts

I am inspired by all sorts of mathematics and am always on the lookout for examples of visual mathematics. If you have some math that could be transformed into jewelry, contact me at *math@hanusadesign.com* and let's talk!

Christopher Hanusa is a professor of mathematics at Queens College of the City University of New York. His research is in algebraic and enumerative combinatorics, and he is an internationally exhibited mathematical artist. He would like to offer readers 15% off any order at his 3D printed mathematical jewelry company Hanusa Design, using promo code HORIZONTAL through November 15, 2022.