DETERMINANTS IN THE KRONECKER PRODUCT OF MATRICES: THE INCIDENCE MATRIX OF A COMPLETE GRAPH

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ABSTRACT. We investigate the least common multiple of all subdeterminants, $\operatorname{lcmd}(A \otimes B)$, of a Kronecker product of matrices, of which one is an integral matrix A with two columns and the other is the incidence matrix of a complete graph with n vertices. We prove that this quantity is the least common multiple of $\operatorname{lcmd}(A)$ to the power $n-1$ and certain binomial functions of the entries of A.

1. Introduction

In a study of non-attacking placements of chess pieces, Chaiken, Hanusa, and Zaslavsky [1] were led to a quasipolynomial formula that depends in part on the least common multiple of the determinants of all square submatrices of a certain Kronecker product matrix, namely, the Kronecker product of an integral 2×2 matrix A with the incidence matrix of a complete graph. We give a compact expression for the least common multiple of the subdeterminants of this product matrix, generalized to A of order $m \times 2$.

2. Background

Kronecker product. For matrices $A = (a_{ij})_{m \times k}$ and $B = (b_{ij})_{n \times l}$, the Kronecker product $A \otimes B$ is defined to be the $mn \times kl$ block matrix

$$\left(\begin{array}{c:c:c} a_{11}B & \cdots & a_{1k}B \\ \hdashline \vdots & \ddots & \vdots \\ \hdashline a_{m1}B & \cdots & a_{mk}B \end{array}\right).$$

It is known (see [2], for example) that when A and B are square matrices of orders m and n, respectively, then $\det(A \otimes B) = \det(A)^n \det(B)^m$.

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The lcmd **operation.** The quantity we want to compute is $\text{lcmd}(A \otimes B)$, where for an integer matrix M, the notation $\text{lcmd}(M)$ denotes the least common multiple of the determinants of all square submatrices of M. This is a much stronger question, as the matrices A and B are most likely not square and the result depends on all square submatrices of their Kronecker product. We discuss properties of this operation in Section 4, after introducing our main result in Section 3.

Incidence matrix. For a simple graph $G = (V, E)$, the incidence matrix $D(G)$ is a $|V| \times |E|$ matrix with a row corresponding to each vertex in V and a column corresponding to each edge in E. For a column that corresponds to an edge $e = vw$, there are exactly two non-zero entries: one $+1$ and one -1 in the rows corresponding to v and w. The sign assignment is arbitrary. The complete graph K_n is the graph on n vertices $v_1, \ldots, v_n$ with an edge between every pair of vertices. Its incidence matrix has order $n \times \binom{n}{2}$.

Of interest in this article are Kronecker products of the form $A \otimes D(K_n)$.

Example 1. We present an illustrative example that we will revisit in the proof of our main theorem. We consider K_4 to have vertices v_1 through v_4, corresponding to rows 1 through 4 of $D(K_4)$, and edges e_1 through e_6, corresponding to columns 1 through 6 of $D(K_4)$. One of the many incidence matrices for K_4 is the 4×6 matrix

$$D(K_4) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

If A is the 3×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$, we investigate the Kronecker product

$$A \otimes D(K_4) =$$

$$\left(\begin{array}{cccccc:cccccc}
a_{11} & a_{11} & a_{11} & 0 & 0 & 0 & a_{12} & a_{12} & a_{12} & 0 & 0 & 0 \\
-a_{11} & 0 & 0 & a_{11} & a_{11} & 0 & -a_{12} & 0 & 0 & a_{12} & a_{12} & 0 \\
0 & -a_{11} & 0 & -a_{11} & 0 & a_{11} & 0 & -a_{12} & 0 & -a_{12} & 0 & a_{12} \\
0 & 0 & -a_{11} & 0 & -a_{11} & -a_{11} & 0 & 0 & -a_{12} & 0 & -a_{12} & -a_{12} \\
\hdashline
a_{21} & a_{21} & a_{21} & 0 & 0 & 0 & a_{22} & a_{22} & a_{22} & 0 & 0 & 0 \\
-a_{21} & 0 & 0 & a_{21} & a_{21} & 0 & -a_{22} & 0 & 0 & a_{22} & a_{22} & 0 \\
0 & -a_{21} & 0 & -a_{21} & 0 & a_{21} & 0 & -a_{22} & 0 & -a_{22} & 0 & a_{22} \\
0 & 0 & -a_{21} & 0 & -a_{21} & -a_{21} & 0 & 0 & -a_{22} & 0 & -a_{22} & -a_{22} \\
\hdashline
a_{31} & a_{31} & a_{31} & 0 & 0 & 0 & a_{32} & a_{32} & a_{32} & 0 & 0 & 0 \\
-a_{31} & 0 & 0 & a_{31} & a_{31} & 0 & -a_{32} & 0 & 0 & a_{32} & a_{32} & 0 \\
0 & -a_{31} & 0 & -a_{31} & 0 & a_{31} & 0 & -a_{32} & 0 & -a_{32} & 0 & a_{32} \\
0 & 0 & -a_{31} & 0 & -a_{31} & -a_{31} & 0 & 0 & -a_{32} & 0 & -a_{32} & -a_{32}
\end{array}\right).$$

Submatrix notation. Let $A = (a_{ij})$ be an $m \times 2$ matrix; this makes $A \otimes D(K_n)$ an $mn \times n(n-1)$ matrix with non-zero entries $\pm a_{ij}$. We introduce new notation for some matrices that will arise naturally in our theorem. For $i, j \in [m] := \{1, 2, \ldots, m\}$, we write $A^{i,j}$ to represent $\begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix}$. If I is a multisubset of $[m]$, we define a_{Ik} to be the product $\prod_{i \in I} a_{ik}$. If I and J are multisubsets of $[m]$, we define $A^{I,J}$ to be the matrix $\begin{pmatrix} a_{I1} & a_{I2} \\ a_{J1} & a_{J2} \end{pmatrix}$. In this notation,

$$\operatorname{lcmd} A = \operatorname{lcm}\big(\operatorname*{LCM}_{i,k} a_{ik}, \operatorname*{LCM}_{i,j} \det A^{i,j}\big),$$

where LCM denotes the least common multiple of non-zero quantities taken over all indicated pairs of indices.

3. Main Theorem and Main Corollary

Let

$$\mathcal{K}_m := \{(I, J) : I, J \text{ are multisubsets of } [m] \\ \text{such that } |I| = |J| \text{ and } I \cap J = \varnothing\}.$$

Recall that a *subdeterminant* or *minor* of a matrix is the determinant of a square submatrix.

Theorem 2. *Let A be an $m \times 2$ matrix, not identically zero, and $n \geq 1$. The least common multiple of all subdeterminants of $A \otimes D(K_n)$ is*

$$\begin{aligned} &\operatorname{lcmd}\big(A \otimes D(K_n)\big) \\ &\qquad = \operatorname{lcm}\bigg((\operatorname{lcmd} A)^{n-1}, \operatorname*{LCM}_{\mathcal{K}} \Big[\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s}\Big]\bigg), \end{aligned} \tag{1}$$

where $\mathrm{LCM}_{\mathcal{K}}$ *denotes the least common multiple of non-zero quantities taken over all collections* $\mathcal{K} \subseteq \mathcal{K}_m$ *such that* $2\sum_{(I,J)\in\mathcal{K}} |I| \leq n$.

The proof, which is long, is in Section 7 at the end of this article. Although the expression is not as simple as we wanted, we were fortunate to find it; it seems to be a much harder problem to get a similar formula when A has more than two columns.

Note that it is only necessary to take the LCM component over all maximal collections $\mathcal{K}$, that is, collections $\mathcal{K}$ satisfying $\sum |I_s| = \lfloor n/2 \rfloor$.

When understanding the right-hand side of Equation (1), it may be instructive to notice that the LCM factor on the right-hand side divides

$$\prod_{\substack{\text{disjoint } I,J:\\ |I|=|J|=p}} (\det A^{I,J})^{\lfloor n/2p \rfloor},$$

since the largest number of individual $\det A^{I,J}$ factors that may occur for disjoint p-member multisubsets I and J of $[m]$ is $\lfloor n/2p \rfloor$.

When $m = 2$, the only pair of disjoint p-member multisubsets of $[m]$ is $\{1^p\}$ and $\{2^p\}$. From this, we have the following corollary.

Corollary 3. *Let A be a 2×2 matrix, not identically zero, and $n \geq 1$. The least common multiple of all subdeterminants of $A \otimes D(K_n)$ is*

$$\begin{aligned} &\operatorname{lcmd}\big(A \otimes D(K_n)\big) \\ &\qquad = \operatorname{lcm}\Big((\operatorname{lcmd} A)^{n-1}, \operatorname*{LCM}_{p=2}^{\lfloor n/2 \rfloor} \big((a_{11}a_{22})^p - (a_{12}a_{21})^p\big)^{\lfloor n/2p \rfloor}\Big), \end{aligned}$$

where LCM *denotes the least common multiple over the range of p.*

4. Properties of the lcmd Operation

Four kinds of operation on A do not affect the value of $\operatorname{lcmd} A$: permuting rows or columns, duplicating rows or columns, adjoining rows or columns of an identity matrix, and transposition. The first two will not change the value of $\operatorname{lcmd}(A \otimes D(K_n))$. However, the latter two may. According to Corollary 3, transposing a 2×2 matrix A does not alter $\operatorname{lcmd}\big(A \otimes D(K_n)\big)$; but when $m > 2$ that is no longer the case, as Example 2 shows. Adding columns of an identity matrix also may change the l.c.m.d., even when A is 2×2; also see Example 2. However, we may freely adjoin rows of I_2 if A has two columns.

Corollary 4. *Let A be an $m \times 2$ matrix, not identically zero, and $n \geq 1$. Let A' be A with any rows of the 2×2 identity matrix adjoined. Then*

$$\operatorname{lcmd}\big(A' \otimes D(K_n)\big) = \operatorname{lcmd}\big(A \otimes D(K_n)\big).$$

Proof. It suffices to consider the case where A' is A adjoin an $(m+1)^{\text{st}}$ row $\begin{pmatrix}1 & 0\end{pmatrix}$. It is obvious that $\operatorname{lcmd} A' = \operatorname{lcmd} A$; this accounts for the first component of the least common multiple in Equation (1).

For the second component, any $\mathcal{K}$ that appears in the LCM for A also appears for A'. Suppose $\mathcal{K}'$ is a collection that appears only for A'; this implies that in $\mathcal{K}'$ there exist pairs (I_s, J_s) such that $m+1 \in I_s$ (or J_s, but that case is similar). Since $a_{I_s2} = 0$, $\det A^{I_s,J_s} = a_{I_s1}a_{J_t2}$, which is a product of at most $n-1$ elements of A. This, in turn, divides $(\operatorname{lcmd} A)^{n-1}$. We conclude that the right-hand side of Equation (1) is the same for A' as for A. □

We do not know whether or not adjoining a row of the identity matrix to an $m \times l$ matrix A preserves $\operatorname{lcmd}\big(A \otimes D(K_n)\big)$ when $l > 2$. Limited calculations give the impression that this may indeed be true.

5. Examples

We calculate a few examples with matrices A that are related to those needed for the chess-piece problem of [1]. In that kind of problem the matrix of interest is $M \otimes D(K_n)^T$ where M is an $m \times 2$ matrix. Hence, in Theorem 2 we want $A = M^T$, so Theorem 2 applies only when $m \leq 2$.

Example 1. When the chess piece is the bishop, M is the 2×2 symmetric matrix

$$M_B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We apply Corollary 3 with $A = M_B$, noting that $\operatorname{lcmd}(A) = 2$. We get

$$\operatorname{lcmd}\big(A \otimes D(K_n)\big) = \operatorname{lcm}\Big(2^{n-1}, \operatorname*{LCM}_{p=2}^{\lfloor n/2 \rfloor} \big((-1)^p - 1^p\big)^{\lfloor n/2p \rfloor}\Big).$$

The LCM generates powers of 2 no larger than $2^{n/2}$, hence $\operatorname{lcmd}\big(M_B \otimes D(K_n)\big) = 2^{n-1}$.

Example 2. When the chess piece is the queen, M is the 4×2 matrix $M_Q = \begin{pmatrix} I \\ M_B \end{pmatrix}$ with $\operatorname{lcmd}(A) = 2$. Then our matrix $A = M_Q^T = \begin{pmatrix} I & M_B \end{pmatrix}$. Since M_Q^T has four columns Theorem 2 does not apply. In fact, we found that $\operatorname{lcmd}\big(M_Q^T \otimes D(K_4)\big) = 24$, quite different from $\operatorname{lcmd}\big(M_B \otimes D(K_4)\big) = 8$.

However, if we take $A = M_Q$ instead of M_Q^T, Corollary 4 applies; we conclude that $\operatorname{lcmd}\big(M_Q \otimes D(K_n)\big) = \operatorname{lcmd}\big(M_B \otimes D(K_n)\big) = 2^{n-1}$.

Thus, $A = M_Q$ is an example where transposing A changes the value of $\operatorname{lcmd}\big(A \otimes D(K_n)\big)$ dramatically.

Example 3. A more difficult example is the fairy chess piece known as a nightrider, which moves an unlimited distance in the directions of a knight. Here M is the 4×2 matrix

$$M_N = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

We can use Theorem 2 to calculate $\operatorname{lcmd}\big(M_N \otimes D(K_n)\big)$. Since what is needed for the chess problem is $\operatorname{lcmd}\big(M_N^T \otimes D(K_n)\big)$, this example does not help in [1]; nevertheless it makes an interestingly complicated application of Theorem 2.

The submatrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix},$$

with determinants -3, -4, and -5, respectively, lead to the conclusion that $\operatorname{lcmd}(A) = 60$. Every pair (I, J) of disjoint p-member multisubsets of $[4]$ has one of the following seven forms, up to the order of I and J:

$$\begin{gathered} \big(\{1^q\}, \{2^r, 3^s, 4^t\}\big), \quad \big(\{2^r\}, \{1^q, 3^s, 4^t\}\big), \\ \big(\{3^s\}, \{1^q, 2^r, 4^t\}\big), \quad \big(\{4^t\}, \{1^q, 2^r, 3^s\}\big), \\ \big(\{1^q, 2^r\}, \{3^s, 4^t\}\big), \quad \big(\{1^q, 3^s\}, \{2^r, 4^t\}\big), \quad \big(\{1^q, 4^t\}, \{2^r, 3^s\}\big), \end{gathered}$$

where the sum of the exponents in each multisubset is p, and where q, r, s, and t may be zero. It turns out that $\det A^{I,J}$ has the same form in all seven cases: precisely $\pm 2^u(2^{2p-2u} \pm 1)$, where u is a number between 0 and p. Furthermore, every value of u from 0 to p appears and every choice of plus or minus sign appears (except when $u = p$) in $\det A^{I,J}$ for some choice of (I, J). We present two representative examples that support this assertion.

The case of $I = \{1^q\}$ and $J = \{2^r, 3^s, 4^t\}$. Then

$$A^{I,J} = \begin{pmatrix} 1^q & 2^q \\ 2^r 1^s 2^t & 1^r (-2)^s (-1)^t \end{pmatrix},$$

with $q = r + s + t = p$. We can rewrite $\det A^{I,J}$ as $\pm 2^s - 2^{2p-s} = -2^s(2^{2p-2s} \pm 1)$. The only instance in where there is no choice of sign is when $s = p$ and $r = t = 0$, in which case $\det A^{I,J}$ simplifies to either 0 or -2^{p+1}.

The case of $I = \{1^q, 2^r\}$ and $J = \{3^s, 4^t\}$. Then

$$A^{I,J} = \begin{pmatrix} 1^q 2^r & 2^q 1^r \\ 1^s 2^t & (-2)^s(-1)^t \end{pmatrix},$$

where $q + r = s + t = p$. For this choice of I and J, $\det A^{I,J} = (-1)^p 2^{r+s} - 2^{2p-r-s}$.

Since every $\det A^{I,J}$ has the same form, and at most $\lfloor p/2n \rfloor$ factors of type $(2^{2p-2u} \pm 1)$ may occur at the same time, the LCM in Equation (1) is exactly

$$\operatorname*{LCM}_{\mathcal{K}} \left(\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s} \right) = 2^N \operatorname*{LCM}_{\substack{1\le p\le n/2 \\ 0\le u\le p-1}} (2^{2p-2u} \pm 1)^{\lfloor n/2p \rfloor},$$

for some $N \le n$. We conclude that

$$\operatorname{lcmd}\big(A \otimes D(K_n)\big) = \operatorname{lcm}(60^{n-1}, \operatorname*{LCM}_{\substack{1\le p\le n/2 \\ 0\le u\le p-1}} (2^{2p-2u} \pm 1)^{\lfloor n/2p \rfloor}).$$

As a sample of the type of answer we get, when $n = 8$ this expression is

$$\begin{aligned} &\operatorname{lcmd}\big(A \otimes D(K_8)\big) \\ &\quad = \operatorname{lcm}(60^7, (4 \pm 1)^{\lfloor 8/2 \rfloor}, (16 \pm 1)^{\lfloor 8/4 \rfloor}, (64 \pm 1)^{\lfloor 8/6 \rfloor}, (256 \pm 1)^{\lfloor 8/8 \rfloor}) \\ &\quad = 60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257. \end{aligned}$$

The first few values of n give the following numbers:

n	$\operatorname{lcmd}\big(A \otimes D(K_8)\big)$	(factored)
2	60	60^1
3	3600	60^2
4	3672000	$60^3 \cdot 17$
5	220320000	$60^4 \cdot 17$
6	1202947200000	$60^5 \cdot 7 \cdot 13 \cdot 17$
7	72176832000000	$60^6 \cdot 7 \cdot 13 \cdot 17$
8	18920434740480000000	$60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
9	1135226084428800000000	$60^8 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
10	952295753183943168000000000	$60^9 \cdot 7 \cdot 11 \cdot 13 \cdot 17^2 \cdot 31 \cdot 41 \cdot 257$

6. Remarks

We hope to determine in the future whether $\operatorname{lcmd}(A \otimes B)$ has a simple form for arbitrary matrices A and B. Our limited experimental data suggests this may be difficult. However, we think at least some generalization of Theorem 2 is possible.

We would like to understand, at minimum, why the theorem as stated fails when $B = D(K_n)$ and A has more than two columns.

Another direction worth investigating is the number-theoretic aspects of Theorem 2.

7. Proof of the Main Theorem

During the proof we refer from time to time to Example 1, which will give a concrete illustration of the many steps. We assume a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, and a_{32} are non-zero constants.

7.1. **Calculating the determinant of a submatrix.** Consider an $l \times l$ submatrix N of $A \otimes D(K_n)$. We wish to evaluate the determinant of N and show that it divides the right-hand side of Equation (1). We need consider only matrices N whose determinant is not zero, since a matrix with $\det N = 0$ has no effect on the least common multiple.

Since $D(K_n)$ is constructed from a graph, we will analyze N from a graphic perspective. The matrix N is a choice of l rows and l columns from $A \otimes D(K_n)$. This corresponds to a choice of l vertices and l edges from K_n where we are allowed to choose up to m copies of each vertex and up to two copies of an edge. Another way to say this is that we are choosing m subsets of $V(K_n)$, say V_1 through V_m, and two subsets of $E(K_n)$, say E_1 and E_2, with the property that $\sum_{i=1}^{m} |V_i| = \sum_{k=1}^{2} |E_k| = l$. From this point of view, if a row in N is taken from the first n rows of $A \otimes D(K_n)$, we are placing the corresponding vertex of $V(K_n)$ in V_1, and so on, up through a row from the last n rows of $A \otimes D(K_n)$, which corresponds to a vertex in V_m. We will say that the copy of v in V_i is the i^{th} *copy of* v and the copy of e in E_k is the k^{th} *copy of* e.

The order of N satisfies $l \leq 2n-2$ because, if N had $2n-1$ columns of $A \otimes D(K_n)$, then at least one edge set, E_1 or E_2, would contain n edges from K_n. The columns corresponding to these edges would form a dependent set of columns in N, making $\det N = 0$.

In our illustrative example, choose the submatrix N consisting of rows 1, 5, 7, and 10 and columns 1, 4, 7, and 8. Then N is the matrix

$$N = \begin{pmatrix} a_{11} & 0 & a_{12} & a_{12} \\ a_{21} & 0 & a_{22} & a_{22} \\ 0 & -a_{21} & 0 & -a_{22} \\ -a_{31} & a_{31} & -a_{32} & 0 \end{pmatrix},$$

and in the notation above, $V_1 = \{v_1\}$, $V_2 = \{v_1, v_3\}$, $V_3 = \{v_2\}$, $E_1 = \{e_1, e_4\}$, and $E_2 = \{e_1, e_2\}$.

Returning to the proof, within this framework we will now perform elementary matrix operations on N in order to make its determinant easier to calculate. We call the resulting matrix the *simplified matrix* *of* N. Each copy of a vertex v has a row in N associated with it; two rows corresponding to two copies of the same vertex contain the same entries except for the different multipliers a_{ik}. For example, if v is a vertex in both V_1 and V_2, then there is a row corresponding to the first copy with multipliers a_{11} and a_{12} and a row corresponding to the second copy with the same entries multiplied by a_{21} and a_{22}.

There cannot be a vertex in three or more vertex sets since then the corresponding rows of N would be linearly dependent and $\det N$ would be zero.

When there is a vertex in exactly two vertex sets V_i and V_j corresponding to two rows R_i and R_j in N, we perform the following operations depending on the multipliers a_{i1}, a_{i2}, a_{j1}, and a_{j2}. We first notice that $\det A^{i,j} = a_{i1}a_{j2} - a_{i2}a_{j1}$ is non-zero; otherwise, the rows R_i and R_j would be linearly dependent in N and $\det N = 0$. Therefore either both a_{i1} and a_{j2} or both a_{i2} and a_{j1} are non-zero. In the former case, let us add $-a_{j1}/a_{i1}$ times R_i to R_j in order to zero out the entries corresponding to edges in E_1. The multipliers of entries in R_j corresponding to edges in E_2 are now all $\det A^{i,j}/a_{i1}$. Similarly, we can zero out the entries in R_i corresponding to edges in E_2. Lastly, factor out $\det A^{i,j}/a_{j2}a_{i1}$ from R_j. If on the other hand, either multiplier a_{i1} or a_{j2} is zero, then reverse the roles of i and j in the preceding argument. These manipulations ensure that the multiplier of every non-zero entry in N that corresponds to an i^{th} vertex and a k^{th} edge is a_{ik}.

The appearance of a denominator, $a_{i1}a_{j2}$, in the factor $\det A^{i,j}/a_{j2}a_{i1}$ is merely an artifact of the construction; we could have cancelled it by factoring out a_{i1} in row i and a_{j2} in row j. However, if we had done this, the entries of the matrix would no longer be of the form a_{ik}, $-a_{ik}$, and 0, which would make the record-keeping in the coming arguments more tedious.

In our illustrative example, because v_1 is a member of both V_1 and V_2, we perform row operations on the rows of N corresponding to v_1 to yield the simplified matrix of N:

$$N_{\text{simplified}} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & 0 & a_{22} & a_{22} \\ 0 & -a_{21} & 0 & -a_{22} \\ -a_{31} & a_{31} & -a_{32} & 0 \end{pmatrix}.$$

The determinants of N and $N_{\text{simplified}}$ are related by

$$\det N = \frac{\det A^{1,2}}{a_{11}a_{22}} \det N_{\text{simplified}}.$$

The denominator $a_{11}a_{22}$ would disappear if we had chosen to factor out the a_{11} in the first row and the a_{22} in the second row of $N_{\text{simplified}}$.

Returning to the proof, we assert that the simplified matrix of N has no more that two non-zero entries in any column. For a column e corresponding to an edge $e = vw$ in K_n, each of v and w is either in one vertex set V_i or in two vertex sets V_i and V_j. If the vertex corresponds to two rows in N, the above manipulations ensure that there is only one copy of the vertex that has a non-zero multiplier in the column. Another important quality of this simplification is that if a vertex is in more than one vertex set, then every edge incident with one instance of this repeated vertex is now in the same edge set; more precisely, if $v \in V_i \cap V_j$, then every edge incident with the i^{th} copy of v is in E_1 and every edge incident with the j^{th} copy is in E_2, or vice versa.

Since we are assuming $\det N \neq 0$, N has at least one non-zero entry in each column or row. If a row (or column) has exactly one non-zero entry, we can reduce the determinant by expanding in that row (or column). This contributes that non-zero entry as a factor in the determinant. After reducing repeatedly in this way, we arrive at a matrix where each column has exactly two non-zero entries, and each row has at least two non-zero entries. This implies that every row has exactly two non-zero entries as well. After interchanging the necessary columns and rows and possibly multiplying columns by -1, the structure of what we will call the *reduced matrix of* N is a block diagonal matrix where each block B is a weighted incidence matrix of a cycle, such as

$$\begin{pmatrix} y_1 & 0 & 0 & 0 & 0 & -z_6 \\ -z_1 & y_2 & 0 & 0 & 0 & 0 \\ 0 & -z_2 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -z_3 & y_4 & 0 & 0 \\ 0 & 0 & 0 & -z_4 & y_5 & 0 \\ 0 & 0 & 0 & 0 & -z_5 & y_6 \end{pmatrix}.$$

The determinant of a $p \times p$ matrix of this type is $y_1 \cdots y_p - z_1 \cdots z_p$. Therefore, we can write the determinant of N as the product of powers of entries of A, powers of $\det A^{i,j}$, and binomials of this form.

In our illustrative example, we simplify the determinant of $N_{\text{simplified}}$ by expanding in the first row (contributing a factor of a_{11}), and we perform row and column operations to find the reduced matrix of N to be

$$N_{\text{reduced}} = \begin{pmatrix} a_{21} & 0 & -a_{22} \\ -a_{31} & a_{32} & 0 \\ 0 & -a_{22} & a_{22} \end{pmatrix},$$

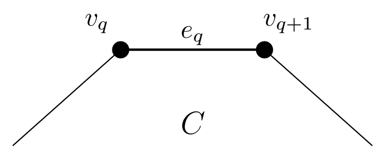

FIGURE 1. An edge $e_q = v_q v_{q+1}$ in the cycle C generated by block B. When $v_q \in V_i$, $v_{q+1} \in V_j$, and $e_q \in E_k$, the contributions y_q and z_q to $\det B$ are a_{ik} and a_{jk}, respectively.

whose determinant is $a_{21}a_{32}a_{22} - a_{31}a_{22}a_{22}$.

Returning to the proof, the entries y_q and z_q are the variables a_{ik}, depending on in which vertex sets the rows lie and in which edge sets the columns lie. If the vertices of K_n corresponding to the rows in B are labeled v_1 through v_p, this block of the block matrix corresponds to traversing the closed walk $C = v_1 v_2 \dots v_p v_1$ in K_n (in this direction). As a result of the form of the simplified matrix of N, for a column that corresponds to an edge $e_q = v_q v_{q+1}$ in E_k traversed from the vertex v_q in vertex set V_i to the vertex v_{q+1} in vertex set V_j, the entry y_q is a_{ik} and the entry z_q is a_{jk}. (See Figure 1.) Therefore each block B in the block diagonal matrix contributes

$$\det B = \prod_{\substack{e = v_q v_{q+1} \in C \\ e \in E_k, v_q \in V_i}} a_{ik} - \prod_{\substack{e = v_q v_{q+1} \in C \\ e \in E_k, v_{q+1} \in V_j}} a_{jk} \tag{2}$$

for some closed walk C in G, whose length is p.

In our illustrative example, N_{reduced} is the incidence matrix of the closed walk

$$v_3 \xrightarrow{e_4} v_2 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_3,$$

where vertex v_3 is from V_2, vertex v_2 is from V_3, and vertex v_3 is from V_2. Moreover, edge e_4 is from E_1, and edges e_1 and e_2 are from E_2. Because we are working with a concrete example, we have not relabeled the vertices as we did in the preceding paragraph.

Returning to the proof, we can simplify this expression by analyzing exactly what the a_{ik} and a_{jk} are. Suppose that two consecutive edges e_{q-1} and e_q in C are in the same edge set E_k, and suppose that the vertex v_q that these edges share is in V_i. (See Figure 2.) In this case, both entries z_{q-1} and y_q are a_{ik}, which can then be factored out of each product in Equation (2).

A particular case to mention is when the cycle C contains a vertex that has multiple copies in N (not necessarily both in C). In this case,

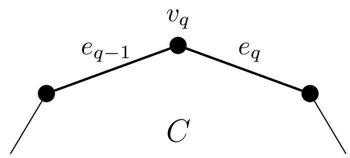

FIGURE 2. Two consecutive edges e_{q-1} and e_q, both incident with vertex v_q in the cycle C generated by block B. When both edges are members of the same edge set E_k and v_q is a member of V_i, the contributions z_{q-1} and y_q are both a_{ik}, allowing this multiplier to be factored out of Equation (2).

the edges of C incident with this repeated vertex are both from the same edge set, as mentioned earlier. After factoring out a multiplier for each pair of adjacent edges in the same edge set, all that remains inside the products in Equation (2) is the contributions of multipliers from vertices where the incident edges are from different edge sets.

More precisely, when following the closed walk, let I be the multiset of indices i such that the walk C passes from an edge in E_2 to an edge in E_1 at a vertex in V_i. Similarly, let J be the multiset of indices j such that C passes from an edge in E_1 to an edge in E_2 at a vertex in V_j. Then what remains inside the products in Equation (2) after factoring out common multipliers is exactly

$$\det A^{I,J} = \prod_{i\in I} a_{i1} \prod_{j\in J} a_{j2} - \prod_{j\in J} a_{j2} \prod_{i\in I} a_{i2}.$$

There is one final simplifying step. Consider a value i occurring in both I and J. In this case, we can factor $a_{i1}a_{i2}$ out of both terms. This implies that the determinant of each block B of the block diagonal matrix is of the form

$$\pm\Bigg(\prod_{i,k} a_{ik}^{s_{ik}}\Bigg) \det A^{I,J}, \tag{3}$$

where the exponents $s_{i,k}$ are non-negative integers, I and J are disjoint subsets of $[m]$ of the same cardinality, and $2|I| + \sum_{i,k} s_{ik} = p$ because the degree of $\det B$ is the order of B. Notice that when $|I| = |J| = 1$ (say $I = \{i\}$ and $J = \{j\}$), the factor $\det A^{I,J}$ equals $\det A^{i,j}$. Combining contributions from the simplification and reduction processes and from all blocks, we have the formula

$$\det N = \pm \prod_{i,j} (\det A^{i,j})^{|V_i \cap V_j|} \prod_{i,k} a_{ik}^{S_{ik}} \prod_{B} \det A^{I_B,J_B}, \tag{4}$$

for some non-negative exponents S_{ik}. We note that

$$\sum 2|V_i \cap V_j| + \sum S_{ik} + \sum |I_B| = l.$$

In the cycle in our illustrative example, vertex v_3 (in V_2) transitions from an edge in E_2 to an edge in E_1 and vertex v_2 (in V_3) transitions from an edge in E_1 to an edge in E_2. This implies that $I = \{2\}$ and $J = \{3\}$. Vertex v_1 originally occurred in the two vertex sets V_1 and V_2; this implies that we can factor out the corresponding multiplier, a_{22}. Indeed, the determinant of N_{reduced} is $a_{21}a_{32}a_{22} - a_{22}^2 a_{31} = a_{22} \det A^{2,3}$. Through these calculations we see that

$$\det N = \left(\frac{\det A^{1,2}}{a_{11}a_{22}} \right) a_{11}a_{22} \det A^{2,3} = \det A^{1,2} \det A^{2,3}.$$

7.2. **The subdeterminant divides the formula.** We now verify that the product in Equation (4) divides the right-hand side of Equation (1). The exponents S_{ik} can be no larger than n because there are only n rows with entries a_{ik} in $A \otimes D(K_n)$, so the expansion of the determinant, as a polynomial in the variable a_{ik}, has degree at most n. Furthermore, it is not possible for the exponent of a_{ik} to be n. The only way this might occur is if N were to contain in V_i all n vertices of G and at least n edges of E_k incident with the vertices of V_i. The corresponding set of columns is a dependent set of columns in N (because the rank of $D(K_n)$ is $n-1$), which would make $\det N = 0$. Therefore, $\det N$ contributes no more than $n-1$ factors of any a_{ik} to any term of $\operatorname{lcmd}(A \otimes D(K_n))$.

Now let us examine the exponents of factors of the form $\det A^{I,J}$ that may divide $\det N$. Such factors may arise either upon the conversion of N to the simplified matrix of N if $|I| = |J| = 1$, or from a block of the reduced matrix as in Equation (3) if $|I| = |J| \geq 1$.

The factors that arise in simplification come from duplicate pairs of vertices: every duplicated vertex v leads to a factor $\det A^{i,j}$ where $v \in V_i \cap V_j$ (this is apparent in Equation (4)). The total number of factors $\det A^{i,j}$ arising from simplification is $\sum_{\{i,j\}} |V_i \cap V_j| = d$, the number of duplicated vertices, which is not more than $n-1$ since $2d \leq l \leq 2(n-1)$. Since each such factor divides $\operatorname{lcmd} A$, their product divides $(\operatorname{lcmd} A)^{n-1}$, the first component of Equation (1).

The factors $\det A^{I_B,J_B}$ from blocks B of the reduced matrix arise from simple vertices—those which are not duplicated among the rows of N. I_B and J_B are multisets of indices of vertex sets V_i containing simple vertices, $I_B \cap J_B = \varnothing$, and $\sum_B (|I_B| + |J_B|) \leq c$, the number of simple vertices, since a simple vertex appears in only one block. As

$c \leq n$, $\sum_B(|I_B| + |J_B|) \leq n$. Thus, the product of the corresponding determinants $\det A^{I_B,J_B}$ is one product in the $\mathrm{LCM}_{\mathcal{K}}$ component of Equation (1).

7.3. **The formula is best possible.** We have shown that for every matrix N, $\det N$ divides the right-hand side of Equation (1). We now show that there exist graphs that attain the claimed powers of factors. Consider the path of length $n-1$, $P = v_1 v_2 \cdots v_n$, as a subgraph of K_n. Create the $(2n-2) \times (2n-2)$ submatrix N of $A \otimes D(K_n)$ with rows corresponding to both an i^{th} copy and a j^{th} copy of vertices v_1 through v_{n-1} and columns corresponding to two copies of every edge in P. Then

$$N = \left(\begin{array}{cccc:cccc} a_{i1} & 0 & 0 & 0 & a_{i2} & 0 & 0 & 0 \\ -a_{i1} & a_{i1} & 0 & 0 & -a_{i2} & a_{i2} & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & -a_{i1} & a_{i1} & 0 & 0 & -a_{i2} & a_{i2} \\ \hdashline a_{j1} & 0 & 0 & 0 & a_{j2} & 0 & 0 & 0 \\ -a_{j1} & a_{j1} & 0 & 0 & -a_{j2} & a_{j2} & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & -a_{j1} & a_{j1} & 0 & 0 & -a_{j2} & a_{j2} \end{array}\right),$$

with determinant $(\det A^{i,j})^{n-1}$. The four quadrants of N are $(n-1) \times (n-1)$ submatrices of $A \otimes D(K_n)$ with determinants a_{i1}^{n-1}, a_{i2}^{n-1}, a_{j1}^{n-1}, and a_{j2}^{n-1}, respectively.

We show that, for every collection $\mathcal{K} = \{(I_s, J_s)\}) \subseteq \mathcal{K}_m$ satisfying $2\sum |I_s| \leq n$, there is a submatrix N of $A \otimes D(K_n)$ with determinant $\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s}$. For each s, starting with $s = 1$, choose $W_s \subseteq V(K_n)$ to consist of the lowest-numbered unused $n_s = 2|I_s|$ vertices. Thus, $W_s = \{v_{2k+1}, \ldots, v_{2k+2n_s}\}$. Take edges $v_{i-1}v_i$ for $2k+1 < i \leq 2k+2n_s$ and $v_{2k+1}v_{2k+2n_s}$. This creates a cycle C_s if $|I_s| > 1$ and an edge e_s if $|I_s| = 1$. For a cycle C_s, place each odd-indexed vertex of W_s into a vertex set V_i for every $i \in I_s$ and each even-indexed vertex into a vertex set V_j for every $j \in J_s$. For an edge e_s corresponding to $I_s = \{i\}$ and $J_s = \{j\}$, place both vertices of W_s in V_i and V_j. Place all edges of the form $v_{2m-1}v_{2m}$ into E_1 and all edges of the form $v_{2m}v_{2m+1}$ and $v_{2k+1}v_{2k+2n_s}$ into E_2. Note that this puts e_s into both E_1 and E_2.

The submatrix N of $A \otimes D(K_n)$ that arises from placing the vertices in numerical order and the edges in cyclic order along C_s is the block-diagonal matrix where each block N_s is a $2|I_s| \times 2|I_s|$ matrix of the form

$$\begin{pmatrix} a_{i_1 1} & 0 & 0 & \cdots & 0 & a_{i_1 2} \\ -a_{j_1 1} & -a_{j_1 2} & 0 & \cdots & 0 & 0 \\ 0 & a_{i_2 2} & a_{i_2 1} & & & 0 \\ 0 & 0 & -a_{j_2 1} & -a_{j_2 2} & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & & -a_{j_l 1} & -a_{j_l 2} \end{pmatrix}$$

if C_s is a cycle and

$$\begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix}$$

if e_s is an edge. The determinant of N_s is exactly $\det A^{I_s,J_s}$, so the determinant of N is $\prod_{(I_s,J_s)\in\mathcal{K}} \det A^{I_s,J_s}$, as desired. □

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