Connectivity

Definition. A graph G is **connected** if for every pair of vertices a and b in G, there is a **path from a to b in G**.

That is, there exists a sequence of distinct vertices $v_0, v_1, \ldots, v_k$ such that $v_0 = a$, $v_k = b$, and $v_{i-1}v_i$ is an edge of G for all i, $1 \leq i \leq k$.

Lemma A. IF there is a path from vertex a to vertex b in G
and a path from vertex b to vertex c in G,
THEN there is a path from vertex a to vertex c in G.

Proof. By hypothesis,

- There exist paths $P : av_1v_2 \cdots v_kb$ and $Q : bw_1w_2 \cdots w_lc$ in G.
- If all the vertices are distinct, path R :
- If not all vertices are distinct, then choose the *first* vertex v_p in P that is also a vertex w_q in Q.

Lemmas A and B

Lemma B. Let G be a connected graph. Suppose that G contains a cycle C and e is an edge of C. The graph $H = G \setminus e$ is connected.

Proof. Let v and w be two vertices of H .
We need to show that there is a path from v to w in H.

Because G is connected, there exists a path $P : v \to w$ in G.
If P does not pass through e, then ______________________________.

If P does pass through $e = xy$, break up P.
Define $P_1 : v \to x$, $P_2 : y \to w$, both paths in H.
We can write the cycle C as $C = xz_1z_2 \cdots z_kyx$.
Therefore, there is a path $Q : x \to y = xz_1z_2 \cdots z_ky$ in H.

Apply Lemma A to show there is a path from v to w in H.

Connectivity and edges

Theorem 1.3.1. If G is a connected graph with p vertices and q edges, then $p \leq q + 1$.

Proof. Induction on the number of edges of G.

- **Base Case.** If G is connected and has fewer than three edges, then G equals either:

- **Inductive Step.**
  Inductive hypothesis:
  $p \leq q + 1$ holds for all connected graphs with $k \geq 3$ edges.

  We want to show:
  $p \leq q + 1$ holds for all connected graphs with

Break into cases, depending on whether G contains a cycle:

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Connectivity and edges

- **Case 1.** There is a cycle C in G.

Use Lemma B. After removing an edge from C, the resulting graph H is connected...

- **Case 2.** There is no cycle in G.

Find a path P in G that can not be extended.

Claim: The endpoints of P, a and b, are leaves of G.

Remove a and its incident edge to form a new graph H.
Apply the inductive hypothesis to H?

□

★ Important Induction Item: Always **remove** edges. ★

Trees and forests

Definition. A **tree** is a connected graph that contains no cycles.
Definition. A **forest** is a graph that contains no cycle.

These definitions imply: (Fill in the blanks)

1. Every connected component of a forest ____________________.
2. A connected forest ____________________.
3. A subgraph of a forest ____________________.
4. A subgraph of a tree ____________________.
5. Every tree is a forest.

Trees are the smallest connected graphs; the following theorems show this and help classify graphs which are trees.

Thm 1.3.2, 1.3.3: Let G be a connected graph with p vertices and q edges. Then,

$$G \text{ is a tree} \iff p = q + 1.$$

Thm 1.3.5. G is a tree iff there exists exactly one path between each pair of vertices.

Proof of Theorem 1.3.3

Thm 1.3.2, 1.3.3: Let G be a connected graph with p vertices and q edges. Then, G is a tree $\iff p = q + 1$.

Proof. ($\Rightarrow$) Use reasoning like Theorem 1.3.1.
(Remove leaves one by one.)

($\Leftarrow$) Proof by contradiction.

Suppose that G is connected and not a tree. Want to show: $p \neq q + 1$.

A graph that is connected and is not a tree ______________________.

By Lemma B, remove an edge from this cycle to find a graph H with ___ vertices and ___ edges.

Theorem 1.3.1 applied to H implies that $p \leq (q - 1) + 1$, so $p \leq q$.

Therefore $p \neq q + 1$.

Proof of Theorem 1.3.5

Thm 1.3.5. G is a tree iff there exists exactly one path between each pair of vertices.

($\Rightarrow$) Suppose that G is a tree. Then G is connected, so for all $v_1, v_2 \in V$, there exists at least one path between v_1 and v_2. Suppose that there are two paths, $P_1 = v_1 u_1 u_2 \cdots u_n v_2$ and $P_2 = v_1 w_1 w_2 \cdots w_m v_2$.

($\Leftarrow$) Suppose G is not a tree.
Either (a) G **is not connected** or (b) G **contains a cycle**.

(a) There exist two vertices v_1 and v_2 with no path between them.
(b) For v_1, v_2 in a cycle, there exist two paths between v_1 and v_2.

In both cases, it is not the case that between each pair of vertices, there exists exactly one path.

Related theorems

Definition. A **bridge** is an edge e such that its removal disconnects G.

Theorem 2.4.1. Suppose that G is a connected. Then

$$G \text{ is a tree} \iff \text{Every edge of } G \text{ is a bridge.}$$

Proof. ($\Rightarrow$) Let $e = vw$ be the edge of a tree G.
The graph $G \setminus e$ is no longer connected because
we removed from G its one path between v and w.

($\Leftarrow$) Let G be a connected graph with a cycle C.
The removal of any edge in C does not disconnect the graph.

Theorem 3.2.1. A regular graph of even degree has no bridge.

Proof. Let G be a regular graph of even degree with a bridge $e = vw$.