Course Notes

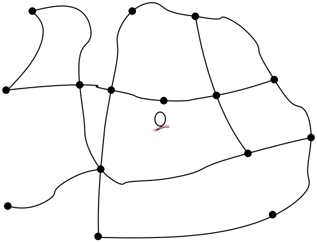
Graph Theory, Spring 2014

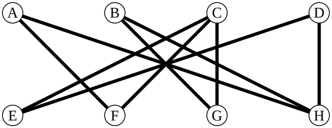
Queens College, Math 634

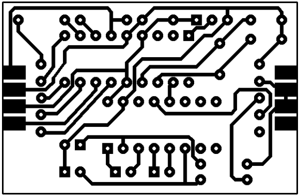
Prof. Christopher Hanusa

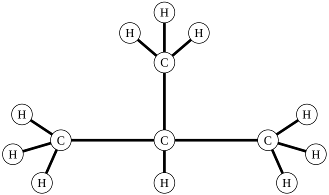
http://qcpages.qc.edu/~chanusa/courses/634sp14/

What is a graph?









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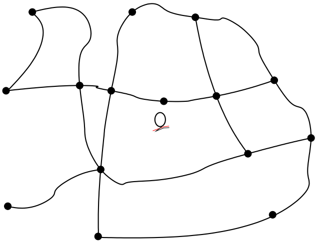

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A “dot” is called a **vertex** (or **node**, **point**, **junction**)
One **vertex** — Two **vertices**.

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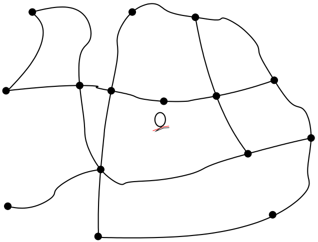

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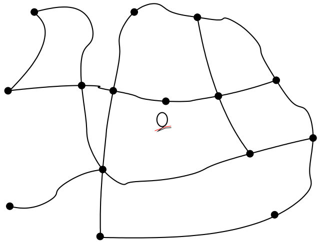

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- Represent each city or intersection as a vertex
- Roads correspond to edges.

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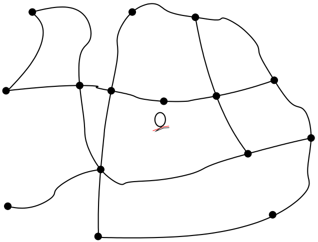

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However, a graph is an abstract concept.

- It doesn’t matter whether the edge is straight or curved.
- All we care about is which vertices are connected.

Concept: Matchings

Suppose that:

Erika likes cherries and dates.
Frank likes apples and cherries.
Greg likes bananas and cherries.
Helen likes apples, bananas, dates.

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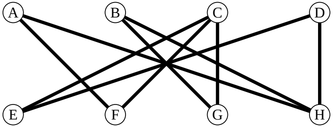

A graph can illustrate these relationships.

- Create one vertex for each person and one vertex for each fruit.
- Create an edge between person vertex v and fruit vertex w if person v likes fruit w.

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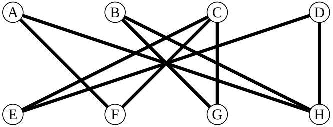

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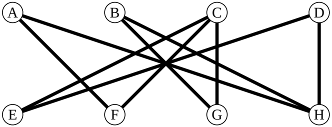

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Related topics: assignments, perfect matchings, counting questions.

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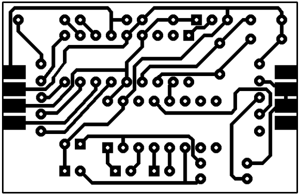

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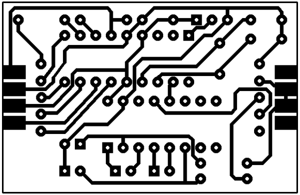
- If not, then how close to being planar is it?



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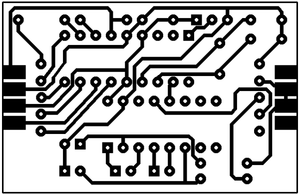

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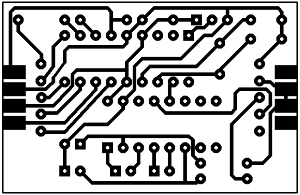

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- Where to drill the holes?
- How to drill them as fast as possible?

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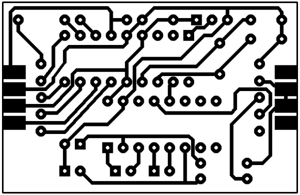

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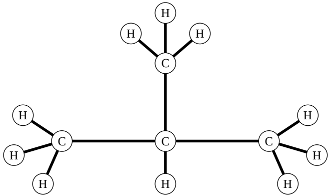
Related topics: Traveling Salesman, computer algorithms, optimization

Chemis-Tree

Graphs are used in Chemistry to draw molecules. (isobutane)



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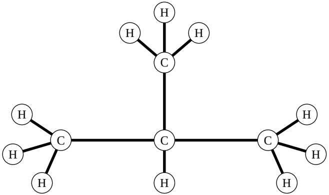

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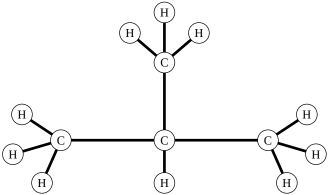

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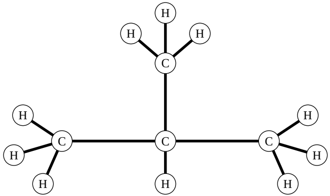
We will work to understand some of their properties.



To do well in this class:

- **Come to class prepared.**
  - Print out and read over course notes.
  - Read sections before class.
- **Form good study groups.**
  - Discuss homework and classwork.
  - Bounce proof ideas around.
  - You will depend on this group.
- **Put in the time.**
  - Three credits = (at least) nine hours / week out of class.
  - Homework stresses key concepts from class; learning takes time.
- **Stay in contact.**
  - If you are confused, ask questions (in class and out).
  - Don't fall behind in coursework or project.
  - I need to understand your concerns.

Mini-assignments due daily; homeworks posted the week before.
Please fill out the notecard: Name, something related to name, picture.

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Example. Let $G = (V, E)$, where
$V = \{v_1, v_2, v_3, v_4\}$,
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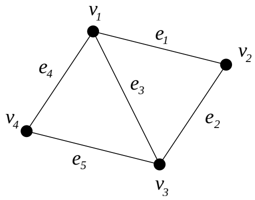

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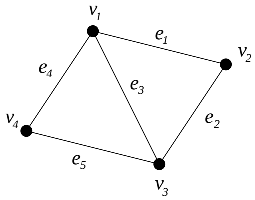

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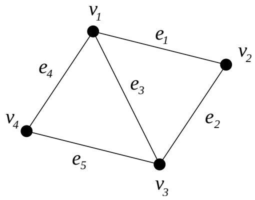

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Notation: # vertices $= |V| =$ __ $=$ __. # edges $= |E| =$ __ $=$ __.

How to talk about a graph

We say v_1 is **adjacent** to v_2 if there is an edge between v_1 and v_2. We also say v_1 and v_2 are **neighbors**.

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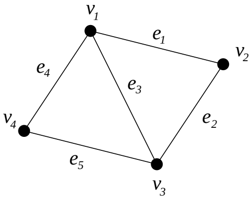
When multiple edges are allowed (but not loops): called **multigraphs**.
When loops (& mult. edge) are allowed: called **pseudographs**.

Degree of a vertex

The **degree** of a vertex v is the number of edges incident with v, and denoted $\deg(v)$.



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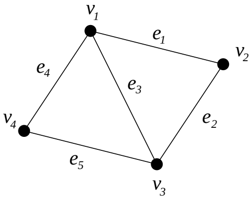

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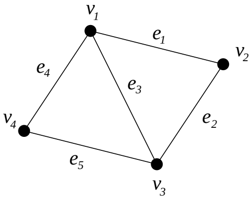

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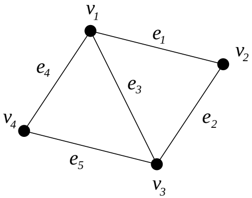

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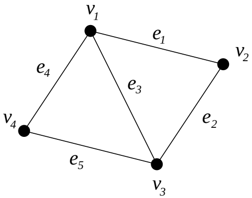

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Q. How many edges in G?

A. $m =$

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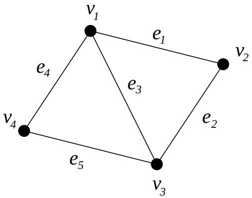

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Q. What is $\deg(v_1) + \deg(v_2) + \deg(v_3) + \deg(v_4)$?

A. $\sum_{v \in V} \deg(v) =$

Q. How many edges in G?

A. $m =$

How are these related?

Degree sum formula

Theorem 1.1.1. $$\sum_{v \in V} \deg(v) = 2m.$$

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Corollary: *The degree sum of a graph is always even.*

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Question. Does every sequence have a simple graph?

Answer.

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Examples: 7765333110 and 6644442

Proof of the Havel–Hakimi algorithm

Notation: Define the degree sequences to be:

$\mathcal{S}_1 = (s, t_1, t_2, \ldots, t_s, d_1, \ldots, d_k).$

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Question: Can this argument work in reverse?

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Game plan:

$$G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots \longrightarrow G_a$$

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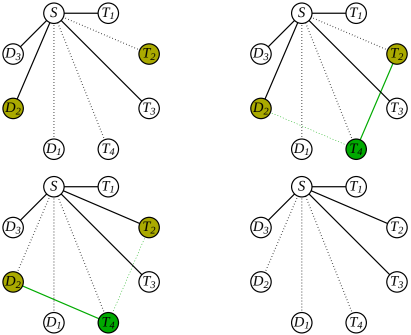
- After some number of iterations, the vertex of highest degree s in G_a will be adjacent to the next s highest degree vertices.
- Peel off vertex S to reveal a graph with degree sequence $\mathcal{S}_2$.

Proof of the Havel–Hakimi algorithm

Vertices $S, T_1, \ldots, T_s, D_1, \ldots, D_k$ have degrees $s, t_1, \ldots, t_s, d_1, \ldots, d_k$.



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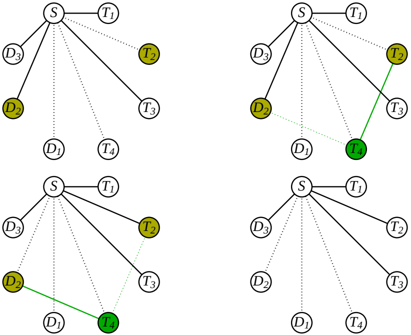

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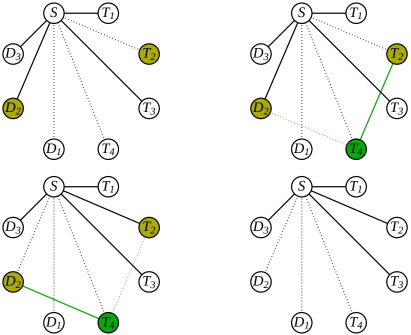

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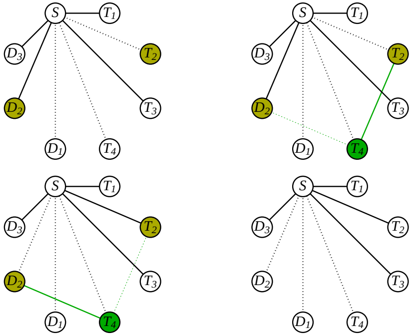

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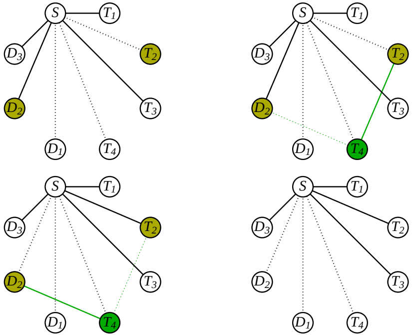

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(c) Replace edges SD_j and T_iV with edges ST_i and D_jV.

(d) The degree sequence of the new graph is the same. (Why?) AND S is now adjacent to more T vertices. (Why?) Repeat as necessary.