

Course Notes

Mathematical Models, Spring 2014

Queens College, Math 245

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What is a model?

A **model** is an object or concept used to represent something else. It converts reality to a form we can comprehend.

- ▶ **Reality:** How to understand the aerodynamics of an airplane?
Model: Use a **model airplane** or a **computer simulation**.
- ▶ **Reality:** Politics flows between left-wing and right-wing ideas.
Model: Think of public opinion as a **pendulum**.

A **mathematical model** is a model involving mathematical concepts.

IN THIS CLASS:

We take **real-world situations** and represent them using mathematics.

- ▶ Model the position of a falling object by **function fitting**.
- ▶ Model people waiting using a **computer simulation**.
- ▶ Model allocating resources using a **system of inequalities**.

Then we must **analyze our models** to determine their applicability.

Why should we model?

As scientists, we want to understand how the world works.



observed real-world
behavior

- ▶ What is happening? (Observation)
- ▶ What are the reasons for the behavior? (Hypothesis)
- ▶ How do we convey that our reasoning is plausible? (“proof”)

— Use the language of mathematics! —

Steps of the Modeling Process

Goal: Understand what is involved in “mathematical modeling”.

First Step: Formulation.

- ▶ **State the question.** If the question is vague, make it precise. If the question is too big, subdivide it into manageable parts.
- ▶ **Identify factors.** Decide which quantities influence the behavior. Determine relationships between the quantities.
[**Important:** *we are introducing* _____.]
- ▶ **Describe mathematically.** Assign each quantity a variable. Represent each relationship with an equation.

Motivating Example: Gravity by Galileo

Example. In Galileo's time, a key question changed from:

Why do objects fall? –to– *How* do objects fall?
 (Philosophical question) (Describe a falling object's velocity)

First Step: **Formulation.**

- ▶ **State the question.** (Precise! What is an answer?)
 What formula describes an object's position as it falls?
- ▶ **Identify factors.** Galileo chose only distance, time, and velocity.
 Other variables?: _____

Simplifying Assumption: Velocity is proportional to the distance fallen.

- ▶ **Describe mathematically.**
 Assign variables. Call **distance** x , **time** t , and **velocity** v .
 Then relationships give equations:
 Velocity and distance are related: $v = \frac{dx}{dt}$.
 And **proportional** means $v = ax$ for some constant a . (Goal?)

Steps of the Modeling Process

- ▶ After the formulation step, we have variables and equations.
- ▶ Do some analysis to develop **mathematical conclusions**.

Second Step: **Mathematical Manipulation**.

This may entail one or more of:

- ▶ Calculations
- ▶ Proving a theorem
- ▶ Solving an equation
- ▶ Other...

In our gravity example,

We have both $v = \frac{dx}{dt}$ and $v = ax$. Set them equal.

This gives the (differential) equation: $\frac{dx}{dt} = ax$.

Solving gives the equation $x(t) = ke^{at}$ for constants a and k .

Something is not quite right...

Steps of the Modeling Process

We have a mathematical conclusion, but does it give a “right answer”?

The *most important* step of the modeling process is:

Third Step: Evaluation.

Translate the results back to the real-world situation and ask questions:

- ▶ Has the model explained the real-world observations?
- ▶ Are the answers we found accurate enough?
- ▶ Were our assumptions good assumptions?
- ▶ What are the strengths and weaknesses of our model?
- ▶ Did we make any mistakes in our mathematical manipulations?

If there are any problems,

- ▶ **Go back** to the First Step, Formulation.
- ▶ Change your assumptions!
- ▶ Start the modeling process over.

Motivating Example: Gravity by Galileo

Third Step: **Evaluation.**

Our mathematical calculations imply that the position of a falling object is $x(t) = ke^{at}$.

In our real-world situation, we can set initial position to be 0. Mathematically, $x(0) = 0$.

This lets us solve for k in our equation:

$$0 = x(0) = ke^{a0} = ke^0 = k.$$

So $k = 0$. Plugging into our equation implies $x(t) = 0$.

In words, this means that **our object stays at rest for all t .**

EPIC FAIL!

Perhaps the proportionality assumption is incorrect?

Motivating Example: Gravity by Galileo

First Step: **Formulation.**

Alternate assumption: The velocity is proportional to the time it has been falling. In particular, the velocity increases by 32 ft/sec.

Mathematically, we have the equations $v = 32t$ and $v = \frac{dx}{dt}$.

Second Step: **Mathematical Manipulation.**

Integrating gives $x(t) = 16t^2 + C$.

Since $x(0) = 0$ we can find $C = 0$.

Therefore an object falling from rest has position $x(t) = 16t^2$.

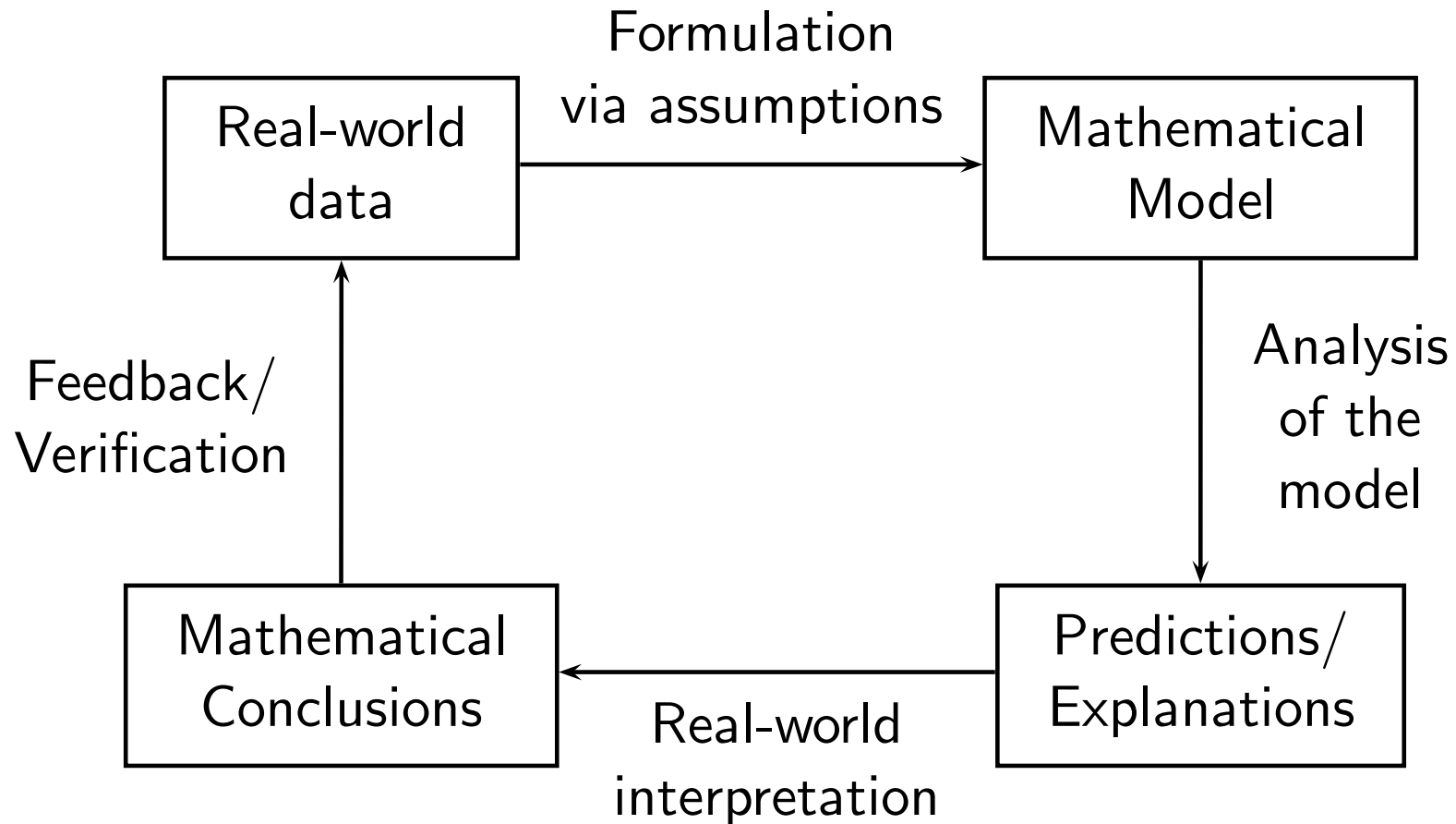
Third Step: **Evaluation.**

This function agrees well with observations in many instances.

(Although not all!)

The Modeling Process

This chart summarizes the modeling process.



To do well in this class:

- ▶ **Come to class prepared.**
 - ▶ Print out and read over course notes.
 - ▶ Read assigned sections before class.
- ▶ **Form good study groups.**
 - ▶ Discuss homework and classwork.
 - ▶ Final project is a group project.
 - ▶ You will depend on this group.
- ▶ **Put in the time.**
 - ▶ Three credits = (at least) nine hours / week out of class.
 - ▶ Homework stresses key concepts from class; learning takes time.
- ▶ **Stay in contact.**
 - ▶ If you are confused, ask questions (in class and out).
 - ▶ Don't fall behind in coursework or project.
 - ▶ I need to understand your concerns.

Homework posted online; Email me by Wednesday + HW1 due Monday.

Choosing a problem statement.

Group Activity. Arrange yourselves into groups of four or five people, with people you **don't know**.

- ▶ Introduce yourself. (your name, where you're from, your major)
- ▶ Fill out **the front of** your notecard:
 - ▶ Write your name. (Stylize if you wish.)
 - ▶ Write a few words related to your name.
 - ▶ *Draw* something in the remaining space.
- ▶ Discuss with your groupmates why you wrote what you wrote.
- ▶ Exchange contact information. (phone / email / other)
- ▶ Work in your group on the worksheet.